

New Generalized Simple Lie Algebras of Cartan Type over a Field with Characteristic 0¹

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Abstract

We construct four new series of generalized simple Lie algebras of Cartan type, using the mixtures of grading operators and down-grading operators. Our results in this paper are further generalizations of those in Osborn's work "New simple infinite-dimensional Lie algebras of characteristic 0."

1 Introduction

The four well-known series of simple Lie algebras of Cartan type were constructed from the derivation algebra of the polynomial algebra in several variables and its subalgebras preserving certain differential forms. The abstract definitions by derivations of generalized Lie algebras of Cartan type appeared in Kac's work [Ka1]. However, it is still a question of how to construct new explicit generalized simple Lie algebras of Cartan type. Kawamoto [K] introduced new simple Lie algebras of generalized Witt type by changing the polynomial algebra in several variables to the group algebra of an additive subgroup of \mathbb{F}^n and considering the Lie subalgebra of its derivation algebras generated by the grading operators, where n is a positive integer and \mathbb{F} is a field with characteristic 0. One can view the operators of taking partial derivatives of the polynomial algebra in several variables as *down-grading operators*. Starting from the derivation subalgebra generated by the grading operators and down-grading operators of the tensor algebra of the group algebra of the direct sum of finite number of additive subgroups of \mathbb{F} with the polynomial algebra in several variables, Osborn [O2] constructed new four series of generalized simple Lie algebras of Cartan type. In [DZ], the authors generalized Kawamoto's work by picking

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out certain subalgebras, whose typical examples are the derivation subalgebra generated by the grading operators and down-grading operators of the tensor algebra of the group algebra of an additive subgroup of \mathbb{F}^n with the polynomial algebra in several variables. The work in [DZ] was also a generalization of Osborn's generalized Witt algebra in [O2]. Passman [P] gave a certain necessary and sufficient condition on derivations for a Lie algebra of generalized Witt type to be simple.

In [O1], Osborn gave a classification of infinite-dimensional simple Novikov algebras with an idempotent element over a field with characteristic 0. We observed in [X3] that the three classes of Osborn's classified simple Novikov algebras can be rewritten in one form by using the sum of a grading operator and a down-grading operator. In other words, these three classes of Novikov algebras can be viewed as one class in terms of our notions. This observation led us to construct in [X3] a much larger class of simple Novikov algebras, including Osborn's classified simple Novikov algebras as very special cases. We also observed that the commutator Lie algebras of Osborn's classified simple Novikov algebras are rank-one simple Lie algebras of generalized Witt type. In fact, by considering the Lie algebras induced by the Hamiltonian operators corresponding to the simple Novikov algebras with an idempotent element, we found in [X3] a new family of infinite-dimensional Lie algebras. Moreover, a new family of infinite-dimensional Lie superalgebras were constructed based on the Hamiltonian superoperators (cf. [X2]) corresponding to the Novikov-Poisson algebras (cf. [X1]) whose Novikov algebraic structures are those classified in [O1]. Recently, we classified in [X4] quadratic conformal superalgebras by certain compatible pairs of a Lie superalgebra and a Novikov superalgebra. Six general constructions of such pairs were given. Moreover, we classified such pairs related to simple Novikov algebras with an idempotent element. As the consequences of this classification, new families of Lie algebras were found. One of the motivations of this paper is to understand the simplicity of these Lie algebras. Another motivation is to understand the simplicity of the Lie algebras generated by our constructed quadratic conformal superalgebras in [X4] (e.g., cf. Section 4.1 in [X5] for the constructions of these Lie algebras).

Our main results in this paper are the constructions and proofs of four new series of generalized simple Lie algebras of Cartan type based on the derivation subalgebra generated by the mixtures of the grading and down-grading operators of the tensor algebra of the group algebra of an additive subgroup of \mathbb{F}^n with the polynomial algebra in several variables. Our Lie algebras of type H contain some non-derivation ingredients.

For the convenience of the reader's understanding this work, below we shall present

the constructions of the four series of simple Lie algebras of Cartan type.

Throughout this paper, let \mathbb{F} be a field with characteristic 0. All the vector spaces are assumed over \mathbb{F} . Denote by \mathbb{Z} the ring of integers and by \mathbb{N} the set of natural numbers $\{0, 1, 2, 3, \dots\}$. We shall always identify \mathbb{Z} with $\mathbb{Z}1_{\mathbb{F}}$ when the context is clear.

Let $\mathcal{A} = \mathbb{F}[t_1, t_2, \dots, t_n]$ be the algebra of polynomials in n variables. A *derivation* ∂ of \mathcal{A} is linear transformation of \mathcal{A} such that

$$\partial(uv) = \partial(u)v + u\partial(v) \quad \text{for } u, v \in \mathcal{A}. \quad (1.1)$$

The typical derivations are $\{\partial_{t_1}, \partial_{t_2}, \dots, \partial_{t_n}\}$, the operators of taking partial derivatives. The space $\text{Der } \mathcal{A}$ of all the derivations of \mathcal{A} forms a Lie algebra. Identifying the elements of \mathcal{A} with their corresponding multiplication operators, we have

$$\text{Der } \mathcal{A} = \sum_{i=1}^n \mathcal{A} \partial_{t_i}, \quad (1.2)$$

which forms a simple Lie algebra. The Lie algebra $\text{Der } \mathcal{A}$ is called a *Witt algebra of rank n* , usually denoted as $\mathcal{W}(n, \mathbb{F})$. The Lie algebra $\mathcal{W}(n, \mathbb{F})$ acts on the Grassmann algebra $\hat{\mathcal{A}}$ of differential forms on \mathcal{A} as follows.

$$\partial(df) = d(\partial(f)), \quad \partial(\omega \wedge \nu) = \partial(\omega) \wedge \nu + \omega \wedge \partial(\nu) \quad (1.3)$$

for $f \in \mathcal{A}$, $\omega, \nu \in \hat{\mathcal{A}}$, $\partial \in \mathcal{W}(n, \mathbb{F})$. Set

$$\mathcal{S}(n, \mathbb{F}) = \{\partial \in \mathcal{W}(n, \mathbb{F}) \mid \partial(dt_1 \wedge dt_2 \wedge \dots \wedge dt_n) = 0\}. \quad (1.4)$$

Assume that $n = 2k$ is an even integer. Define

$$\mathcal{H}(2k, \mathbb{F}) = \{\partial \in \mathcal{W}(n, \mathbb{F}) \mid \partial(\sum_{i=1}^k dt_i \wedge dt_{k+i}) = 0\}. \quad (1.5)$$

Suppose that $n = 2k + 1$ is an odd integer. We let

$$\begin{aligned} \mathcal{K}(2k + 1, \mathbb{F}) &= \{\partial \in \mathcal{W}(n, \mathbb{F}) \mid \partial(dt_{2k+1} + \sum_{i=1}^k (t_i dt_{k+i} - t_{k+i} dt_i)) \\ &\in \mathcal{A}(dt_{2k+1} + \sum_{i=1}^k (t_i dt_{k+i} - t_{k+i} dt_i))\}. \end{aligned} \quad (1.6)$$

The subspaces $\mathcal{S}(n, \mathbb{F})$, $\mathcal{H}(2k, \mathbb{F})$ and $\mathcal{K}(2k + 1, \mathbb{F})$ form simple Lie subalgebras, which are called the Lie algebras of *Special type*, *Hamiltonian type* and *Contact type*, respectively. For convenience, we simply call the Lie algebras $\mathcal{W}(n, \mathbb{F})$, $\mathcal{S}(n, \mathbb{F})$, $\mathcal{H}(2k, \mathbb{F})$ and $\mathcal{K}(2k + 1, \mathbb{F})$ the Lie algebras of *type W*, *S*, *H* and *K*, respectively. It can be observed that the

simplicity of the Lie algebras of type S, H and K is determined only by their concrete presentations of elements (e.g., cf. [SF]) and does not have any direct relations with their defining differential forms. This essentially gives us rooms to generalize these algebras.

We shall process our constructions and proofs section by section according to the Cartan type W, S, H and K.

2 Algebras of Type W

In this section, we shall construct and prove a new class of generalized simple Lie algebras of Witt type.

Given $m, n \in \mathbb{Z}$ with $m < n$, we shall use the following notion

$$\overline{m, n} = \{m, m+1, m+2, \dots, n\} \quad (2.1)$$

throughout this paper. We also treat $\overline{m, n} = \emptyset$ when $m > n$.

Definition 2.1. Let \mathcal{A} be a commutative associative algebra with an identity $1_{\mathcal{A}}$ and let $\{\partial_1, \dots, \partial_n\}$ be n linearly independent and mutually commutative derivations of \mathcal{A} such that

$$\mathbb{F}1_{\mathcal{A}} + \sum_{p=1}^n \partial_p(\mathcal{A}) = \mathcal{A}, \quad \{u \in \mathcal{A} \mid \partial_1(u) = \dots = \partial_n(u) = 0\} = \mathbb{F}1_{\mathcal{A}}. \quad (2.2)$$

Identify the elements of \mathcal{A} with their corresponding multiplication operators. The following subspace of derivations

$$\mathcal{W} = \sum_{i=1}^n \mathcal{A} \partial_i \quad (2.3)$$

forms a Lie subalgebra of the Lie algebras of all the derivations of \mathcal{A} . We call \mathcal{W} a *generalized Lie algebra of Witt type*. In fact, its Lie bracket is given by

$$\left[\sum_{p=1}^n u_p \partial_p, \sum_{q=1}^n v_q \partial_q \right] = \sum_{p,q=1}^n (u_p \partial_p(v_q) - \partial_p(u_q) v_p) \partial_q \quad (2.4)$$

for $u_p, v_q \in \mathcal{A}$.

Next we shall give the specific construction of our generalized simple Lie algebras of Witt type. Let n be a positive integer. Pick

$$\mathcal{J}_p \in \{\{0\}, \mathbb{N}\} \quad \text{for } p \in \overline{1, n}. \quad (2.5)$$

Let Γ be a torsion-free abelian group and let $\{\varphi_p \mid p \in \overline{1, n}\}$ be n additive group homomorphisms from Γ to \mathbb{F} such that

$$\bigcap_{p \neq q \in \overline{1, n}} \ker \varphi_q \setminus \ker \varphi_p \neq \emptyset \quad \text{if } \mathcal{J}_p = \{0\} \text{ for } p \in \overline{1, n}, \quad (2.6)$$

$$\bigcap_{p=1}^n \ker \varphi_p = \{0\}. \quad (2.7)$$

Condition (2.7) implies that Γ is isomorphic to an additive subgroup of \mathbb{F}^n .

Set

$$\vec{\mathcal{J}} = \mathcal{J}_1 \times \mathcal{J}_2 \times \cdots \times \mathcal{J}_n, \quad (2.8)$$

where the addition of $\vec{\mathcal{J}}$ is defined componentwise. Moreover, we denote:

$$i_{[p]} = (0, \dots, \overset{p}{i}, 0, \dots, 0) \quad \text{for } i \in \mathcal{J}_p. \quad (2.9)$$

Let \mathcal{A} be a vector space with a basis

$$\{x^{\alpha, \vec{i}} \mid (\alpha, \vec{i}) \in \Gamma \times \vec{\mathcal{J}}\}. \quad (2.10)$$

We define an algebraic operation \cdot on \mathcal{A} by

$$x^{\alpha, \vec{i}} \cdot x^{\beta, \vec{j}} = x^{\alpha+\beta, \vec{i}+\vec{j}} \quad \text{for } (\alpha, \vec{i}), (\beta, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}. \quad (2.11)$$

Then \mathcal{A} forms an associative algebra with an identity element $x^{0, \vec{0}}$, which will be simply denoted by 1 in the rest of this paper. Moreover, we define $\partial_{\varphi_p}, \hat{\partial}_p \in \text{End } \mathcal{A}$ for $p \in \overline{1, n}$ by:

$$\partial_{\varphi_p}(x^{\alpha, \vec{i}}) = \varphi_p(\alpha)x^{\alpha, \vec{i}}, \quad \hat{\partial}_p(x^{\alpha, \vec{i}}) = i_p x^{\alpha, \vec{i}-1_{[p]}} \quad (2.12)$$

for $(\alpha, \vec{i}) \in \Gamma \times \vec{\mathcal{J}}$ (cf. (2.9)), where we adopt the convention that if a notion is not defined but technically appears in an expression, we always treat it as zero; for instance, $x^{\alpha, -1_{[1]}} = 0$ for any $\alpha \in \Gamma$. Then $\{\partial_{\varphi_p}, \hat{\partial}_q \mid p, q \in \overline{1, n}\}$ are derivations of \mathcal{A} . Furthermore,

$$\hat{\partial}_p^{i_p+1}(x^{\alpha, \vec{i}}) = 0 \quad \text{for } p \in \overline{1, n}, (\alpha, \vec{i}) \in \Gamma \times \vec{\mathcal{J}}. \quad (2.13)$$

Thus ∂_{φ_p} is a grading operator and $\hat{\partial}_p$ is a down-grading operator. We set

$$\partial_p = \partial_{\varphi_p} + \hat{\partial}_p \quad \text{for } p \in \overline{1, n} \quad (2.14)$$

and define $(\mathcal{W}, [\cdot, \cdot])$ by (2.3) and (2.4). It can be verified that (2.2) holds.

Theorem 2.2. *The Lie algebra $(\mathcal{W}, [\cdot, \cdot])$ is simple.*

Proof. It can be verified that our construction ingredients satisfy Passman's simplicity conditions of the Lie algebras of Witt type (cf. [P]). \square

Example. Let $n = n_1 + n_2$ with $n_1, n_2 \in \mathbb{N}$. We pick

$$\mathcal{J}_p = \{0\}, \quad \mathcal{J}_{n_1+q} = \mathbb{N} \quad \text{for } p \in \overline{1, n_1}, q \in \overline{1, n_2} \quad (2.15)$$

(cf. (2.5)). Let $m \in \mathbb{N}$ be such that $n_1 \leq m \leq n$ and define

$$\zeta_p(\alpha_1, \alpha_2, \dots, \alpha_m) = \alpha_p \quad \text{for } p \in \overline{1, m}, (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{F}^m. \quad (2.16)$$

Now we take Γ to be an additive subgroup of \mathbb{F}^m such that

$$\Gamma \supset \{(j_1, \dots, j_{n_1}, 0, \dots, 0) \mid j_1, \dots, j_{n_1} \in \mathbb{Z}\} \quad (2.17)$$

and define

$$\varphi_p = \zeta_p|_{\Gamma}, \quad \varphi_q \equiv 0 \quad \text{for } p \in \overline{1, m}, q \in \overline{m+1, n}. \quad (2.18)$$

Then $\varphi_1, \dots, \varphi_n$ are additive group homomorphisms satisfying (2.6) and (2.7). For instance, we can take

$$\Gamma = \sum_{j=1}^k ((j/k, j/k, \dots, j/k) + \mathbb{Z}^m) \quad (2.19)$$

for any positive integer k . When $n_1 = 0$, $m = n$ and $\Gamma = \mathbb{Z}^n$,

$$\mathcal{A} = \mathbb{F}[t_i, t_{n+i}, t_i^{-1} \mid i \in \overline{1, n}] \quad (2.20)$$

(cf. (2.10) and (2.11)) and

$$\partial_i = t_i \partial_{t_i} + \partial_{t_{n+i}} \quad \text{for } i \in \overline{1, n}. \quad (2.21)$$

3 Algebras of Type S

In this section, we shall construct and prove a new class of generalized simple Lie algebras of Special type.

We shall use the same notations as in last section. For convenience, we denote

$$x_1^\alpha = x^{\alpha, \vec{0}}, \quad x_2^{\vec{i}} = x^{0, \vec{i}} \quad \text{for } \alpha \in \Gamma, \vec{i} \in \vec{\mathcal{J}}. \quad (3.1)$$

We shall restrict to a special case of $(\Gamma, \vec{\varphi})$. Let $\{\Delta_p \mid p \in \overline{1, n}\}$ be n additive subgroups of \mathbb{F} . Set

$$\Gamma = \Delta_1 \times \Delta_2 \times \dots \times \Delta_n, \quad (3.2)$$

where the addition on $\vec{\Delta}$ is defined componentwise. We use $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ to denote an element in Γ with $\alpha_p \in \Delta_p$. Moreover, for $p \in \overline{1, n}$, we take φ_p as

$$\varphi_p(\vec{\alpha}) = \alpha_p \quad \text{for } \vec{\alpha} \in \Gamma. \quad (3.3)$$

Furthermore, we denote

$$\alpha_{[p]} = (0, \dots, \overset{p}{\alpha}, 0, \dots, 0) \quad \text{for } \alpha \in \Delta_p. \quad (3.4)$$

Now Condition (2.6) becomes

$$\Delta_p \neq \{0\} \quad \text{if } \mathcal{J}_p = \{0\} \text{ for } p \in \overline{1, n} \quad (3.5)$$

and Condition (2.7) holds automatically.

Recall the algebra (\mathcal{A}, \cdot) defined by (2.10), (2.11) and $\{\partial_1, \dots, \partial_n\}$ defined by (2.12) and (2.14). Moreover, we assume that $n \geq 2$.

Let $\vec{\rho}, \vec{\sigma} \in \Gamma$ be two fixed elements. We set

$$D_p = x_1^{(\sigma_p)_{[p]}} \partial_p \quad \text{for } p \in \overline{1, n} \quad (3.6)$$

(cf. (3.1) and (3.4)). Then $\{D_1, \dots, D_n\}$ are linearly independent and mutually commutative derivations of \mathcal{A} satisfying (2.2). Moreover, we set

$$D_{p,q}(u) = x_1^{\vec{\rho}}(D_q(u)D_p - D_p(u)D_q) \quad (3.7)$$

for $p, q \in \overline{1, n}$, $u \in \mathcal{A}$. By (3.1) in [O2],

$$\begin{aligned} & [D_{p,q}(u), D_{r,s}(v)] \\ = & D_{p,s}(x_1^{\vec{\rho}} D_q(u) D_r(v)) + D_{q,r}(x_1^{\vec{\rho}} D_p(u) D_s(v)) \\ & - D_{p,r}(x_1^{\vec{\rho}} D_q(u) D_s(v)) - D_{q,s}(x_1^{\vec{\rho}} D_p(u) D_r(v)) \end{aligned} \quad (3.8)$$

for $p, q, r, s \in \overline{1, n}$ and $u, v \in \mathcal{A}$. We define:

$$\mathcal{S} = \text{span} \{D_{p,q}(u) \mid p, q \in \overline{1, n}, u \in \mathcal{A}\} \subset \mathcal{W}. \quad (3.9)$$

In the spirit of (3.8), \mathcal{S} forms a Lie subalgebra of \mathcal{W} , which we call a *generalized Lie algebra of Special type*. The following result was proved by Osborn [O2].

Proposition 3.1. *The Lie algebra $\mathcal{S}^{(1)} = [\mathcal{S}, \mathcal{S}]$ is a simple Lie algebra when $\Delta_p = \{0\}$ or $\mathcal{J}_p = \{0\}$ for each $p \in \overline{1, n}$.*

Our main theorem in this section is:

Theorem 3.2. *The Lie algebra \mathcal{S} is simple if $\Delta_p \neq \{0\}$ and $\mathcal{J}_p \neq \{0\}$ for some $p \in \overline{1, n}$.*

We shall prove the theorem by establishing several lemmas. For convenience, we can assume:

$$\Delta_1 \neq \{0\}, \quad \mathcal{J}_1 \neq \{0\}. \quad (3.10)$$

Set

$$\iota_{p,q} = (\rho_p + \sigma_p)_{[p]} + (\rho_q + \sigma_q)_{[q]}, \quad \vec{\gamma}_{p,q} = (\gamma_p)_{[p]} + (\gamma_q)_{[q]} \quad \text{for } \vec{\gamma} \in \Gamma \quad (3.11)$$

(cf. (3.4)).

For any $(\vec{\alpha}, \vec{i})$ and $(\vec{\beta}, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}$, (3.8) implies the following identity

$$\begin{aligned} & [D_{p,q}(x^{\vec{\alpha}, \vec{i}}), D_{p,q}(x^{\vec{\beta}, \vec{j}})] \\ = & (\alpha_q \beta_p - \alpha_p \beta_q) D_{p,q}(x^{\vec{\alpha} + \vec{\beta} + \vec{\rho} + \vec{\sigma}_{p,q}, \vec{i} + \vec{j}}) + (i_q j_p - i_p j_q) D_{p,q}(x^{\vec{\alpha} + \vec{\beta} + \vec{\rho} + \vec{\sigma}_{p,q}, \vec{i} + \vec{j} - 1_{[p]} - 1_{[q]}}) \\ & + (\alpha_q j_p - \beta_q i_p) D_{p,q}(x^{\vec{\alpha} + \vec{\beta} + \vec{\rho} + \vec{\sigma}_{p,q}, \vec{i} + \vec{j} - 1_{[p]}}) \\ & + (\beta_p i_q - \alpha_p j_q) D_{p,q}(x^{\vec{\alpha} + \vec{\beta} + \vec{\rho} + \vec{\sigma}_{p,q}, \vec{i} + \vec{j} - 1_{[q]}}) \end{aligned} \quad (3.12)$$

(cf. (2.9)).

In the rest of this section, we let I be a nonzero ideal of \mathcal{S} . Moreover, by reindexing if necessary, we can assume

$$\Delta_p \neq \{0\}, \quad \Delta_q = \{0\} \quad \text{for } p, q \in \overline{1, n}, \quad p \leq m, \quad q > m, \quad (3.13)$$

where $m \in \overline{1, n}$. Obviously, $m \geq 1$ by our assumption (3.10). A nonzero element

$$u = \sum_{p=1}^n \sum_{\vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}} a_{p, \vec{\alpha}, \vec{i}} x^{\vec{\alpha}, \vec{i}} \partial_p \in \mathcal{W} \quad (3.14)$$

is called a *homogeneous element of degree k* if

$$\sum_{p=1}^m i_p = k \quad \text{whenever } a_{q, \vec{\alpha}, \vec{i}} \neq 0 \quad \text{for } q \in \overline{1, n}, \quad (\vec{\alpha}, \vec{i}) \in \Gamma \times \vec{\mathcal{J}}. \quad (3.15)$$

For a homogeneous element $u \in \mathcal{W}$, we denote its degree by $\wp(u)$. The degree of 0 is defined as -1 . Furthermore, for $k \in \mathbb{N}$, we define:

$$\mathcal{W}_k = \text{span} \{ \text{homogeneous element } u \in \mathcal{W} \mid \wp(u) \leq k \}. \quad (3.16)$$

The leading term u_{ld} of a nonzero element $u \in \mathcal{W}$ is a nonzero homogeneous element defined by

$$u = u_{ld} + u' \quad \text{with } u' \in \mathcal{W}_{k-1}, \quad \wp(u_{ld}) = k \quad \text{for some } k \in \mathbb{N}. \quad (3.17)$$

Lemma 3.3. *The subspace $I_{1,2} = I \cap (\mathcal{A}\partial_1 + \mathcal{A}\partial_2) \neq \{0\}$.*

Proof. For any $u \in \mathcal{W}$, we write

$$u = u' + u^* \quad \text{with } u' = \sum_{(\vec{\alpha}, \vec{i}) \in \Gamma \times \vec{\mathcal{J}}} x^{\vec{\alpha}, \vec{i}} (a_{1, \vec{\alpha}, \vec{i}} \partial_1 + a_{2, \vec{\alpha}, \vec{i}} \partial_2), \quad u^* \in \sum_{p=3}^n \mathcal{A} \partial_p, \quad (3.18)$$

where $a_{1,\vec{\alpha},\vec{i}}, a_{2,\vec{\alpha},\vec{i}} \in \mathbb{F}$. Moreover, we write the leading term:

$$(u^*)_{ld} = \sum_{p=3}^n \sum_{(\vec{\alpha},\vec{i}) \in \Gamma \times \vec{\mathcal{J}}} b_{p,\vec{\alpha},\vec{i}} x^{\vec{\alpha},\vec{i}} \partial_p \quad (3.19)$$

with all $b_{p,\vec{\alpha},\vec{i}} \in \mathbb{F}$ and define

$$\iota(u) = |\{(p, \vec{\alpha}, \vec{i}) \mid b_{p,\vec{\alpha},\vec{i}} \neq 0, 2 < p \in \overline{1, n}, \vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}\}|. \quad (3.20)$$

Set

$$\hat{k} = \min \{\wp((u^*)_{ld}) \mid 0 \neq u \in I\}, \quad (3.21)$$

where $\wp((u^*)_{ld})$ is the degree of $(u^*)_{ld}$ defined by (3.15). Our statement in the lemma is equivalent to $\hat{k} = -1$. Assume that $\hat{k} \geq 0$. We define

$$\iota = \min \{\iota(u) \mid 0 \neq u \in I, \wp((u^*)_{ld}) = \hat{k}\}. \quad (3.22)$$

If $\iota = 0$, then we get a contradiction to (3.20). Now we assume $\iota > 0$. Pick any $u \in I$ such that $\wp((u^*)_{ld}) = \hat{k}$ and $\iota(u) = \iota$. We write u as in (3.18) and (3.19).

Case 1. $\Delta_2 \neq \{0\}$.

Subcase 1. There exist $b_{p,\vec{\alpha},\vec{i}} \neq 0$ and $b_{q,\vec{\beta},\vec{j}} \neq 0$ such that $\alpha_1 = 0$ and $\beta_1 \neq 0$.

Pick any $0 \neq \gamma \in \Delta_2$. Note

$$D_{1,2}(x_1^{\gamma[2]}) = \gamma x_1^{\vec{\rho} + \vec{\sigma}_{1,2} + \gamma[2]} \partial_1. \quad (3.23)$$

Then

$$v = [D_{1,2}(x_1^{\gamma[2]}), u] \in I, \quad \wp((v^*)_{ld}) = \hat{k}, \quad 0 < \iota(v) < \iota(u) = \iota, \quad (3.24)$$

which contradicts (3.20).

Subcase 2. All $\alpha_1 = 0$ for $b_{p,\vec{\alpha},\vec{i}} \neq 0$.

Suppose that $b_{q,\vec{\beta},\vec{j}} \neq 0$. We shall use (3.18) and (3.19). First we assume that $\Delta_q \neq \{0\}$.

For any $0 \neq \gamma \in \Delta_q$, we have:

$$\gamma^{-1}[u^*, D_{1,q}(x_1^{\gamma[q]})] \equiv \sum_{\vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}} (\gamma + \rho_q + \sigma_q) b_{q,\vec{\alpha},\vec{i}} x^{\vec{\alpha} + \gamma + \vec{\rho} + \vec{\sigma}_{1,q}, \vec{i}} \partial_1 \pmod{\mathcal{W}_{k-1}}. \quad (3.25)$$

Since the coefficients of $x^{\vec{\alpha},\vec{i}} \partial_1$ and $x^{\vec{\alpha},\vec{i}} \partial_2$ in $\gamma^{-1}[u', D_{1,q}(x_1^{\gamma[q]})]$ (cf. (3.17)) are independent of γ , only finite number of them are nonzero and $|\Delta_q| = \infty$, (3.25) enables us to choose $0 \neq \gamma \in \Delta_q$ such that

$$0 \neq v = \gamma^{-1}[u, D_{1,q}(x_1^{\gamma[q]})] \in I, \quad \wp((v^*)_{ld}) < \hat{k}, \quad (3.26)$$

which contradicts (3.21).

Next we assume $\Delta_q = \{0\}$. Then $\mathcal{J}_q = \mathbb{N}$ by our earlier assumption. For any $0 \neq l \in \mathbb{N}$, we have:

$$l^{-1}[u^*, D_{1,q}(x_2^{l[q]})] \equiv \sum_{\vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}} (l-1)b_{q,\vec{\alpha},\vec{i}} x^{\vec{\alpha} + \vec{\rho} + \vec{\sigma}_1, q, \vec{i} + (l-2)[q]} \partial_1 \pmod{\mathcal{W}_{k-1}}. \quad (3.27)$$

Since the coefficients of $x^{\vec{\alpha}, \vec{i}} \partial_1$ and $x^{\vec{\alpha}, \vec{i}} \partial_2$ in $l^{-1}[u', D_{1,q}(x_2^{l[q]})]$ are independent of l and only finite number of them are nonzero, (3.27) enables us to choose $0 \neq l \in \mathbb{N}$ such that

$$0 \neq v = l^{-1}[u, D_{1,q}(x_2^{l[q]})] \in I, \quad \wp((v^*)_{ld}) < \hat{k}, \quad (3.28)$$

which contradicts (3.21).

We have the same conclusion if $\alpha_2 = 0$ for some $b_{p,\vec{\alpha},\vec{i}} \neq 0$.

Subcase 3. All $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, whenever $b_{p,\vec{\alpha},\vec{i}} \neq 0$.

Assume that $b_{q,\vec{\beta},\vec{j}} \neq 0$. If $\beta_1 \neq -(\rho_1 + \sigma_1)$, then

$$v = [D_{1,2}(x_1^{-(\beta_1 + \rho_1 + \sigma_1)[1]}), u] \in I, \quad \wp((v^*)_{ld}) = \hat{k}, \quad \iota(v) = \iota(u) \quad (3.29)$$

and v has a term

$$(\beta_1 + \rho_1 + \sigma_1)\beta_2 b_{q,\vec{\beta},\vec{j}} x^{\vec{\beta} - (\beta_1)[1] + \vec{\rho} - (\rho_1)[1] + (\sigma_2)[2], \vec{j}} \partial_q \quad \text{if } \Delta_2 \neq \{0\}. \quad (3.30)$$

Replacing u by v , we go back to the above Subcases 1 and 2. If $\beta_2 + \rho_2 + \sigma_2 = 0$, we choose any $0 \neq \gamma \in \Delta_1$. Replacing u by $[D_{1,2}(x_1^{\gamma[1]}), u]$, we get the same situation as in the above Subcases 1 and 2 with indices 1 and 2 exchanged. Now we assume that $\beta_2 + \rho_2 + \sigma_2 \neq 0$ and $\beta_1 = -(\rho_1 + \sigma_1)$. Pick any $\gamma \in \Delta_1$ such that $\gamma \neq 0$, $-(\rho_1 + \sigma_1)$. We have:

$$v = [D_{1,2}(x_1^{-(\gamma + \rho_1 + \sigma_1)[1]}), [D_{1,2}(x_1^{\gamma[1]}), u]] \in (I \cap \mathcal{W}_k) \setminus \mathcal{W}_{k-1}, \quad \iota(v) = \iota(u). \quad (3.31)$$

Replacing u by v , we again go back to the above Subcases 1 and 2.

In summary, we always get a contradiction if $\Delta_2 \neq \{0\}$.

Case 2. $\Delta_2 = \{0\}$.

According to our assumption, $m = 1$ in this case (cf. (3.13)). Thus $\mathcal{J}_2 = \cdots = \mathcal{J}_n = \mathbb{N}$.

Subcase 1. There exist $b_{p,\vec{\alpha},\vec{i}} \neq 0$ and $b_{q,\vec{\beta},\vec{j}} \neq 0$ such that $\alpha_1 = 0$ and $\beta_1 \neq 0$.

Note

$$D_{1,2}(x_2^{1[2]}) = x_1^{\vec{\rho} + (\sigma_1)[1]} \partial_1. \quad (3.32)$$

$$v = [D_{1,2}(x_2^{1[2]}), u] \in I, \quad \wp((v^*)_{ld}) = \hat{k}, \quad 0 < \iota(v) < \iota(u_{ld}) = \iota, \quad (3.33)$$

which contradicts (3.22).

Subcase 2. All $\alpha_1 = 0$ for $b_{p,\vec{\alpha},\vec{i}} \neq 0$.

This is the same as the situation that $\Delta_q = \{0\}$ in Subcase 2 of Case 1.

We have the same conclusion if $i_2 = 0$ for some $b_{p,\vec{\alpha},\vec{i}} \neq 0$.

Subcase 3. All $\alpha_1 \neq 0$, $i_2 \neq 0$ whenever $b_{p,\vec{\alpha},\vec{i}} \neq 0$.

Assume that $b_{q,\vec{\beta},\vec{j}} \neq 0$. If $\beta_1 \neq -(\rho_1 + \sigma_1)$, then we have (3.29), and v has a term

$$(\beta_1 + \rho_1 + \sigma_1)j_2 b_{q,\vec{\beta},\vec{j}} x^{\vec{\beta} - (\beta_1)_{[1]} + \vec{\rho} - (\rho_1)_{[1]}, \vec{j} - 1_{[2]}} \partial_q. \quad (3.34)$$

Replacing u by v , we go back to the above Subcases 1 and 2. If $j_2 = 1$, we choose any $0 \neq \gamma \in \Delta_1$. Replacing u by $[D_{1,2}(x_1^{\gamma[1]}), u]$, we get the same situation as in the above with indices 1 and 2 exchanged. Now we assume that $j_2 > 1$ and $\beta_1 = -(\rho_1 + \sigma_1)$. Pick any $\gamma \in \Delta_1$ such that $\gamma \neq 0$, $-(\rho_1 + \sigma_1)$. We have (3.31). Thus, we always get a contradiction if $\Delta_2 = \{0\}$.

Therefore, we get a contradiction if $\hat{k} \geq 0$. This completes the proof of the lemma. \square

For $\vec{i} \in \vec{\mathcal{J}}$, we define

$$|\vec{i}|_{1,2} = \begin{cases} i_1 & \text{if } \Delta_2 = \{0\}, \\ i_1 + i_2 & \text{if } \Delta_2 \neq \{0\}. \end{cases} \quad (3.35)$$

Moreover, for

$$u = \sum_{(\alpha,\vec{i}) \in \Gamma \times \vec{\mathcal{J}}} x^{\alpha,\vec{i}} (a_{1,\alpha,\vec{i}} \partial_1 + a_{2,\alpha,\vec{i}} \partial_2) \in \mathcal{W}, \quad (3.36)$$

we let

$$\wp_{1,2}(u) = \max \{ |\vec{i}|_{1,2} \mid a_{1,\alpha,\vec{i}} \neq 0 \text{ or } a_{2,\alpha,\vec{i}} \neq 0 \}. \quad (3.37)$$

In addition, we put $\wp_{1,2}(0) = -1$. Set

$$\mathcal{W}_{1,2}^k = \{u \in \mathcal{A}\partial_1 + \mathcal{A}\partial_2 \mid \wp_{1,2}(u) \leq k\} \quad \text{for } k \in \mathbb{N}. \quad (3.38)$$

Lemma 3.4. *If $(I_{1,2} \cap \mathcal{W}_{1,2}^k) \setminus \mathcal{W}_{1,2}^{k-1} \neq \emptyset$ for some $k \in \mathbb{N}$, then there exists an element $u \in (I_{1,2} \cap \mathcal{W}_{1,2}^k) \setminus \mathcal{W}_{1,2}^{k-1}$ such that*

$$u \equiv \sum_{\vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}} c_{\vec{\alpha},\vec{i}} D_{1,2}(x^{\vec{\alpha},\vec{i}}) \pmod{\mathcal{W}_{1,2}^{k-1}} \quad (3.39)$$

with

$$(\alpha_1, \alpha_2) = (\kappa_1, \kappa_2) \text{ whenever } c_{\vec{\alpha}, \vec{i}} \neq 0 \quad (3.40)$$

if $\Delta_2 \neq 0$, or

$$(\alpha_1, i_2) = (\kappa_1, \varepsilon) \text{ whenever } c_{\vec{\alpha}, \vec{i}} \neq 0 \quad (3.41)$$

if $\Delta_2 = \{0\}$, where $\kappa_p \in \Delta_p$ and $\varepsilon \in \mathbb{N}$.

Proof. Assume $(I_{1,2} \cap \mathcal{W}_{1,2}^k) \setminus \mathcal{W}_{1,2}^{k-1} \neq \{0\}$. For any $u \in (I_{1,2} \cap \mathcal{W}_{1,2}^k) \setminus \mathcal{W}_{1,2}^{k-1}$, we write

$$u \equiv \sum_{\vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}} a_{\vec{\alpha}, \vec{i}} D_{1,2}(x^{\vec{\alpha}, \vec{i}}) \pmod{\mathcal{W}_{1,2}^{k-1}} \quad (3.42)$$

with

$$D_{1,2}(x^{\vec{\alpha}, \vec{i}}) \in \mathcal{W}_{1,2}^k \setminus \mathcal{W}_{1,2}^{k-1} \quad \text{whenever } a_{\vec{\alpha}, \vec{i}} \neq 0. \quad (3.43)$$

Set

$$\ell(u) = \begin{cases} |\{(\alpha_1, \alpha_2) \mid a_{\vec{\alpha}, \vec{i}} \neq 0\}| & \text{if } \Delta_2 \neq \{0\}, \\ |\{(\alpha_1, i_2) \mid a_{\vec{\alpha}, \vec{i}} \neq 0\}| & \text{if } \Delta_2 = \{0\}. \end{cases} \quad (3.44)$$

Moreover, we let

$$\ell = \min\{\ell(u) \mid u \in (I_{1,2} \cap \mathcal{W}_{1,2}^k) \setminus \mathcal{W}_{1,2}^{k-1}\}. \quad (3.45)$$

We want to prove that $\ell = 1$.

Assume $\ell > 1$. We pick $u \in (I_{1,2} \cap \mathcal{W}_{1,2}^k) \setminus \mathcal{W}_{1,2}^{k-1}$ such that $\ell(u) = \ell$. Write u_{id} by (3.42) and (3.43).

Case 1. $\Delta_2 \neq \{0\}$.

Subcase 1. There exist $a_{\vec{\alpha}, \vec{i}} \neq 0$ and $a_{\vec{\beta}, \vec{j}} \neq 0$ such that $(\beta_1, \beta_2) \notin \mathbb{F}(\alpha_1, \alpha_2)$.

In this situation, we have

$$v = [D_{1,2}(x_1^{\vec{\alpha}}), u] \in (I_{1,2} \cap \mathcal{W}_{1,2}^k) \setminus \mathcal{W}_{1,2}^{k-1}, \quad \ell(v) < \ell(u) \quad (3.46)$$

by (3.11), which contradicts (3.45).

Subcase 2. Assume $a_{\vec{\alpha}, \vec{i}} \neq 0$. If $a_{\vec{\beta}, \vec{j}} \neq 0$, then $(\beta_1, \beta_2) \in \mathbb{F}(\alpha_1, \alpha_2)$.

Since $\ell > 1$, there exists $b_{\vec{\beta}, \vec{j}} \neq 0$ such that $(\beta_1, \beta_2) \neq (\alpha_1, \alpha_2)$. Pick any $(\gamma_1, \gamma_2) \in \Delta_1 \times \Delta_2$ such that $(\gamma_1, \gamma_2), (\gamma_1 + \rho_1 + \sigma_1, \gamma_2 + \rho_2 + \sigma_2) \notin \mathbb{F}(\alpha_1, \alpha_2)$. Then

$$v = [D_{1,2}(x_1^{(\gamma_1)_{[1]} + (\gamma_2)_{[2]}}), u] \in (I_{1,2} \cap \mathcal{W}_{1,2}^k) \setminus \mathcal{W}_{1,2}^{k-1}, \quad \ell(v) = \ell(u). \quad (3.47)$$

Replacing u by this v , we go back to the above Subcase 1.

Case 2. $\Delta_2 = \{0\}$. In this case, $\mathcal{J}_2 = \mathbb{N}$.

Subcase 1. There exist $a_{\vec{\alpha}, \vec{i}} \neq 0$ and $a_{\vec{\beta}, \vec{j}} \neq 0$ such that $(\beta_1, j_2) \notin \mathbb{F}(\alpha_1, i_2)$.

In this situation, we have (3.46) with $x_1^{\vec{\alpha}}$ replaced by $x_1^{\vec{\alpha}, (i_2)_{[2]}}$, which contradicts (3.45).

Subcase 2. Assume $a_{\vec{\alpha}, \vec{i}} \neq 0$. If $a_{\vec{\beta}, \vec{j}} \neq 0$, then $(\beta_1, j_2) \in \mathbb{F}(\alpha_1, i_2)$.

Since $\ell > 1$, there exists $a_{\vec{\beta}, \vec{j}} \neq 0$ such that $(\beta_1, j_2) \neq (\alpha_1, i_2)$. Pick any $(\gamma, l) \in \Delta_1 \times \mathbb{N}$ such that $(\gamma, l), (\gamma + \rho_1 + \sigma_1, l) \notin \mathbb{F}(\alpha_1, i_2)$. Then

$$v = [D_{1,2}(x^{\gamma_{[1]}, l_{[2]}}), u] \in (I_{1,2} \cap \mathcal{W}_{1,2}^k) \setminus \mathcal{W}_{1,2}^{k-1}, \quad \ell(v) = \ell(u) \quad (3.48)$$

by (3.11). Replacing u by this v , we go back to the above Subcase 1.

Therefore, $\ell = 1$, which implies our conclusion in the lemma. \square

Set

$$\tilde{k} = \min\{k \mid I_{1,2} \cap \mathcal{W}_{1,2}^k \neq \{0\}\}, \quad (3.49)$$

which is well-defined by Lemma 3.3.

Lemma 3.5. *There exists an element $u \in I_{1,2}$ such that*

$$u = \sum_{\vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}} d_{\vec{\alpha}, \vec{i}} D_{1,2}(x^{\vec{\alpha}, \vec{i}}) \quad (3.50)$$

with

$$\alpha_1 = 0, \quad i_1 = 0 \quad (\alpha_2, i_2) = (\kappa_2, \varepsilon) \quad \text{whenever } d_{\vec{\alpha}, \vec{i}} \neq 0, \quad (3.51)$$

for any $\kappa_2 \in \Delta_2$ and $\varepsilon \in \mathbb{N}$ such that

$$\varepsilon = 0, \quad \kappa_2 \neq 0, \quad \rho_2 + \sigma_2 \quad \text{if } \Delta_2 \neq \{0\}; \quad \varepsilon \neq 0 \quad \text{if } \Delta_2 = \{0\}. \quad (3.52)$$

Proof. Take u to be an element as in the statement of Lemma 3.4 with $k = \tilde{k}$.

Case 1. $\Delta_2 \neq \{0\}$.

Assume that $\tilde{k} > 0$. If $(\kappa_1, \kappa_2) = (0, 0)$, then $0 \leq \wp_{1,2}([D_{1,2}(x_1^{(\gamma_1)_{[1]} + (\gamma_2)_{[2]}}), u]) < \tilde{k}$ for any $0 \neq \gamma_1 \in \Delta_1$, $0 \neq \gamma_2 \in \Delta_2$, which contradicts (3.49). Assume $(\kappa_1, \kappa_2) \neq (0, 0)$. Note that

$$\begin{aligned} & [D_{1,2}(x_1^{(\kappa_1)_{[1]} + (\kappa_2)_{[2]}}), u] \\ \equiv & \sum_{\vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}} c_{\vec{\alpha}, \vec{i}} [\kappa_2 i_1 D_{1,2}(x^{\vec{\alpha} + \vec{\rho} + \vec{\sigma}_{1,2} + (\kappa_1)_{[1]} + (\kappa_2)_{[2]}, \vec{i} - 1_{[1]}}) \\ & - \kappa_1 i_2 D_{1,2}(x^{\vec{\alpha} + \vec{\rho} + \vec{\sigma}_{1,2} + (\kappa_1)_{[1]} + (\kappa_2)_{[2]}, \vec{i} - 1_{[2]}})] \\ & + \sum_{(\vec{\beta}, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}; \beta_p \neq 2\kappa_p + \sigma_p + \rho_p, p=1 \text{ or } 2} a'_{\vec{\beta}, \vec{j}} D_{1,2}(x^{\vec{\beta}, \vec{j}}) \pmod{\mathcal{W}_{1,2}^{\tilde{k}-2}}. \end{aligned} \quad (3.53)$$

Considering $[D_{1,2}(x_1^{\vec{i}}), u] \in I$ if necessary, we can take any $(\kappa_1, \kappa_2) \neq (\rho_1 + \sigma_1, \rho_2 + \sigma_2)$ without changing \vec{i} for all $c_{\vec{\alpha}, \vec{i}} \neq 0$ in (3.39) by (3.11) and the proof of Proposition 3.6 in [O2] (also cf. (3.60)-(3.67)). Choose u such that $\kappa_1 \neq 0, \kappa_2 = 0$ if $i_2 \neq 0$ and $\kappa_2 \neq 0, \kappa_1 = 0$ otherwise. For such u , $[D_{1,2}(x_1^{(\kappa_1)[1] + (\kappa_2)[2]}), u] \in (I \cap \mathcal{W}_{1,2}^{\tilde{k}-1}) \setminus \mathcal{W}_{1,2}^{\tilde{k}-2}$, which contradicts (3.49). So $\tilde{k} = 0$. Considering $[D_{1,2}(x_1^{\vec{i}}), u] \in I$ we can obtain $\kappa_1 = 0$ and $\kappa_2 \neq 0, \rho_2 + \sigma_2$ by (3.11) and the proof of Proposition 3.6 in [O2] (also cf. (3.60)-(3.67)).

Case 2. $\Delta_2 = \{0\}$.

In this case, $\mathcal{J}_2 = \mathbb{N}$ by (2.6). Assume that $\tilde{k} > 0$. If $(\kappa_1, \varepsilon) = (0, 0)$, then we have $0 \leq \wp_{1,2}([D_{1,2}(x_1^{(\gamma_1)[1], 1[2]}), u]) < \tilde{k}$ for any $0 \neq \gamma_1 \in \Delta_1, 0 \neq l \in \mathbb{N}$, which contradicts (3.49). Assume $(\kappa_1, \varepsilon) \neq (0, 0)$. Considering $[D_{1,2}(x^{\gamma[1], l[2]}), u] \in I$ if necessary, we can take any $(\kappa_1, \varepsilon) \neq (\rho_1 + \sigma_1, 0)$ without changing \vec{i} for all $c_{\vec{\alpha}, \vec{i}} \neq 0$ in (3.39) by (3.11) and the proof of Proposition 3.6 in [O2] (also cf. (3.60)-(3.67)). In particular, we can take $\kappa_1 = 0$ and $\varepsilon \neq 0$. Assume that $i_1 \neq 0$ for some $c_{\vec{\alpha}, \vec{i}} \neq 0$ in (3.39). Since

$$\begin{aligned} & [D_{1,2}(x_2^{1[2]}), u] \\ \equiv & \sum_{\vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}} c_{\vec{\alpha}, \vec{i}} i_1 D_{1,2}(x^{\vec{\alpha} + \vec{\rho} + \vec{\sigma}_{1,2}, \vec{i} - 1[1]}) \\ & + \sum_{(\vec{\beta}, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}; \beta_1 \neq \rho_1 + \sigma_1 \text{ or } j_2 \neq \varepsilon} b_{\vec{\beta}, \vec{j}} D_{1,2}(x^{\vec{\beta}, \vec{j}}) \pmod{\mathcal{W}_{1,2}^{\tilde{k}-2}}, \end{aligned} \quad (3.54)$$

we have $[D_{1,2}(x_2^{1[2]}), u] \in (I \cap \mathcal{W}_{1,2}^{\tilde{k}-1}) \setminus \mathcal{W}_{1,2}^{\tilde{k}-2}$, which contradicts (3.49). Thus $i_1 = 0$. \square

Lemma 3.6. *We have $D_{1,n}(x_1^{(\alpha_1)[1]}) \in I$ for some $0 \neq \alpha_1 \in \Delta_1$.*

Proof. For any $p \in \overline{1, n}$, we define:

$$\Psi_p = \begin{cases} (\Delta_p, 0) & \text{if } \Delta_p \neq \{0\}, \\ (0, \mathbb{N}) & \text{if } \Delta_p = \{0\}. \end{cases} \quad (3.55)$$

For $p > 3$, we define $I_{1,p}$ to be the subset of I of the elements of the form:

$$u = \sum_{\vec{\alpha} \in \Gamma, \vec{i} \in \vec{\mathcal{J}}} a_{\vec{\alpha}, \vec{i}} D_{1,p}(x^{\vec{\alpha}, \vec{i}}) \quad (3.56)$$

with $a_{\vec{\alpha}, \vec{i}} \in \mathbb{F}$ and

$$(\alpha_q, i_q) = (\kappa_q, \varepsilon_q) \quad \text{for } 2 \leq q \leq p-1 \quad \text{whenever } a_{\vec{\alpha}, \vec{i}} \neq 0, \quad (3.57)$$

where

$$(\kappa_q, \varepsilon_q) \in \Psi_q \quad \text{for } 2 \leq q \leq p-1 \quad (3.58)$$

are fixed for each u .

Let u be an element as Lemma 3.5 (cf. (3.50)). By considering $[u, D_{1,2}(x^{\vec{\alpha}, \vec{i}})]$ with $i_1 = 0$, $(\alpha_2, i_2) \in \Psi_2$ (cf. (3.11)) and the proof of Proposition 3.6 in [O2]), we can assume that u satisfies (3.50)-(3.52) and

$$(\alpha_3, i_3) \neq (0, 0) \quad \text{whenever } d_{\vec{\alpha}, \vec{i}} \neq 0. \quad (3.55)$$

Note that such an element $u \in I_{1,3}$. Thus $I_{1,3} \neq \{0\}$. Replacing the index 2 by 3 in the proofs of Lemmas 3.4 and 3.5, we can prove that $I_{1,4} \neq \{0\}$. Continuing this process, we can prove that $I_{1,n} \neq \{0\}$ by induction. Replacing the index 2 by n in the proofs of Lemmas 3.4 and 3.5, we get $u = D_{1,n}(x^{\vec{\alpha}, \vec{i}}) \in I$ with $\alpha_1 = 0$, $i_1 = 0$, $(\alpha_p, i_p) \in \Psi_p$ for $2 \leq p \in \overline{1, n}$ and $(\alpha_n, i_n) \neq (0, 0)$. Repeating considering $[u, D_{1,n}(x^{\vec{\beta}, \vec{j}})]$ with $j_1 = 0$, $(\beta_p, j_p) \in \Psi_p$ for $2 \leq p \in \overline{1, n}$ by (3.11) and the proof of Proposition 3.6 in [O2] (also cf. (3.60)-(3.67)), we can obtain $D_{1,n}(x_1^{(\alpha_1)[1]}) \in I$ for some $0 \neq \alpha_1 \in \Delta_1$. \square

Proof of Theorem 3.2.

We want to prove that $I = \mathcal{S}$. By Lemma 3.6 and reindexing, we can assume that $D_{1,2}(x_1^{(\alpha_1)[1]}) \in I$ for convenience.

Case 1. $\Delta_2 \neq 0$.

For any $\vec{\beta} \in \Gamma$ such that $\beta_2 \neq 0$, we have:

$$I \ni [D_{1,2}(x_1^{\alpha_{[1]}}), D_{1,2}(x_1^{\vec{\beta}})] = -\alpha\beta_2 D_{1,2}(x_1^{\alpha_{[1]} + \vec{\beta} + \vec{\rho} + \vec{\sigma}_{1,2}}) \quad (3.60)$$

by (3.11). So we obtain

$$D_{1,2}(x_1^{\alpha_{[1]} + \vec{\beta} + \vec{\rho} + \vec{\sigma}_{1,2}}) \in I. \quad (3.61)$$

This implies that

$$D_{1,2}(x_1^{\vec{\beta} + \iota_{1,2}}) \in I \quad \text{for } \vec{\beta} \in \Gamma, \beta_2 \neq 0. \quad (3.62)$$

If $\rho_2 + \sigma_2 = 0$, then $\vec{\gamma}_{1,2} - 2\iota_{1,2} - \vec{\beta}_{1,2}$ and $\vec{\beta}_{1,2} + \iota_{1,2}$ are linearly independent for any $\vec{\beta}, \vec{\gamma} \in \Gamma$ such that $\gamma_1 \neq \rho_1 + \sigma_1$, $\gamma_2 = 0$ and $\beta_2 \neq 0$. For such $\vec{\beta}$ and $\vec{\gamma}$, since

$$[D_{1,2}(x_1^{\vec{\gamma} - \vec{\beta} - \vec{\rho} - \vec{\sigma}_{1,2} - \iota_{1,2}}), D_{1,2}(x_1^{\vec{\beta} + \iota_{1,2}})] \in I, \quad (3.54)$$

we have

$$D_{1,2}(x_1^{\vec{\gamma}}) \in I \quad (3.64)$$

by (3.11). If $\rho_2 + \sigma_2 \neq 0$, then $D_{1,2}(x_1^{2\iota_{1,2}}) \in I$ by (3.63). Then for any $\gamma \in \Gamma$ such that $\gamma_1 \neq 0$ and $\gamma_2 = 0$, since

$$[D_{1,2}(x_1^{\vec{\gamma} - 2\iota_{1,2}}), D_{1,2}(x_1^{2\iota_{1,2}})] = 2\gamma_1(\rho_2 + \sigma_2)D_{1,2}(x_1^{\vec{\gamma} + \vec{\rho} + \vec{\sigma}_{1,2}}) \in I, \quad (3.65)$$

we have

$$D_{1,2}(x_1^{\vec{\beta}+\iota_{1,2}}) \in I \quad \text{for } \vec{\beta} \in \Gamma, \beta_1 \neq 0. \quad (3.66)$$

Combining (3.63) and (3.66), we get

$$D_{1,2}(x_1^{\vec{\beta}}) \in I \quad \text{for } \vec{\beta} \in \Gamma, \vec{\beta}_{1,2} \neq \iota_{1,2}. \quad (3.67)$$

For any $\vec{\beta} \in \Gamma$ such that $\beta_1, \beta_2 \neq 0$ and $\vec{\beta}_{1,2} \neq \iota_{1,2}$,

$$[D_{1,2}(x_1^{\vec{\beta},1[1]}), D_{1,2}(x_1^{-\vec{\beta}_{1,2}})] = \beta_2 D_{1,2}(x_1^{\vec{\beta}-\vec{\beta}_{1,2}+\vec{\rho}+\vec{\sigma}_{1,2}}) \in I \quad (3.68)$$

by (3.11). Therefore,

$$D_{1,2}(x_1^{\vec{\beta}}) \in I \quad \text{for any } \vec{\beta} \in \Gamma. \quad (3.69)$$

Case 2. $\Delta_2 = 0$.

According to our assumption, $\mathcal{J}_2 = \mathbb{N}$ in this case. Since for any $\vec{\beta} \in \Gamma$ and $0 < j \in \mathbb{N}$, we have

$$I \ni [D_{1,2}(x_1^{\alpha_{[1]}}), D_{1,2}(x_1^{\vec{\beta},j[2]})] = -\alpha j D_{1,2}(x_1^{\alpha_{[1]}+\vec{\beta}+\vec{\rho}+\vec{\sigma}_{1,2},(j-1)[2]}). \quad (3.70)$$

Thus

$$D_{1,2}(x_1^{\vec{\beta},j[2]}) \in I \quad \text{for any } \vec{\beta} \in \Gamma, j \in \mathbb{N}. \quad (3.71)$$

We now want to prove $I = \mathcal{S}$ by (3.69) and (3.71). Let $2 < r \in \overline{1, n}$. If $\Delta_2 \neq \{0\}$, we have:

$$[D_{1,2}(x_1^{\beta_{[2]}}), D_{2,r}(x_1^{\vec{\gamma}})] = \beta \gamma_2 D_{1,r}(x_1^{\beta_{[2]}+\vec{\gamma}+\vec{\rho}+(2\sigma_2)[2]}) - \beta \gamma_r D_{1,2}(x_1^{\beta_{[2]}+\vec{\gamma}+\vec{\rho}+\vec{\sigma}_{2,r}}) \in I \quad (3.72)$$

for any $\beta \in \Delta_2$ and $\vec{\gamma} \in \Gamma$ by (3.7). Thus by (3.69),

$$D_{1,r}(x_1^{\vec{\alpha}}) \in I \quad \text{for any } \vec{\alpha} \in \Gamma \quad (3.73)$$

if $\Delta_2 \neq \{0\}$. If $\Delta_2 = \{0\}$, then $\mathcal{J}_2 = \mathbb{N}$ and (3.71) holds. In this case, we have:

$$[D_{1,2}(x_2^{1[2]}), D_{2,r}(x_1^{\vec{\beta},1[2]})] = D_{1,r}(x_1^{\vec{\beta}+\vec{\rho}}) - \beta_r D_{1,2}(x_1^{\vec{\beta}+\vec{\rho}+(\sigma_r)_{[r]},1[2]}) \in I \quad (3.70)$$

for any $\vec{\beta} \in \Gamma$. Thus (3.73) holds again.

Now we let $1 < r \in \overline{1, n}$. If $\mathcal{J}_r \neq \{0\}$, then for any $\vec{\beta} \in \Gamma$ and $\vec{j} \in \mathcal{J}$, we have:

$$[D_{1,r}(x_2^{1[r]}), D_{1,r}(x_1^{\vec{\beta},\vec{j}})] = \beta_1 D_{1,r}(x_1^{\vec{\beta}+\vec{\rho}+\vec{\sigma}_{1,r},\vec{j}}) + j_1 D_{1,r}(x_1^{\vec{\beta}+\vec{\rho}+\vec{\sigma}_{1,r},\vec{j}-1[1]}) \in I \quad (3.75)$$

by (3.11) and (3.71). Moreover, by (3.75) and induction on j_1 , we can prove that

$$D_{1,r}(x^{\vec{\beta}, \vec{j}}) \in I \quad \text{for any } \vec{\beta} \in \Gamma, \vec{j} \in \vec{\mathcal{J}}. \quad (3.76)$$

If $\mathcal{J}_r = \{0\}$, then $\Delta_r \neq \{0\}$ by our assumption. Exchanging the positions of 1 and r , we can prove (3.76) because $D_{1,r}(x^{\alpha_{[r]}}) \in I$ for $\alpha \in \Delta_r$ by (3.65) and (3.73).

Now for any $1 < r, s \in \overline{1, n}$, $\vec{\beta}, \vec{\gamma} \in \Gamma$ and $\vec{j} \in \vec{\mathcal{J}}$, we have:

$$\begin{aligned} [D_{1,r}(x^{\vec{\beta}, \vec{j}}), D_{1,s}(x_1^{\vec{\gamma}})] &\equiv \beta_1 \gamma_1 D_{s,r}(x^{\vec{\beta} + \vec{\gamma} + \vec{\rho} + (2\sigma_1)_{[1]}, \vec{j}}) \\ &\quad + j_1 \gamma_1 D_{s,r}(x^{\vec{\beta} + \vec{\gamma} + \vec{\rho} + (2\sigma_1)_{[1]}, \vec{j} - 1_{[1]}}) \pmod{I}. \end{aligned} \quad (3.77)$$

by (3.8). By (3.77) and induction on j_1 , we can prove that

$$D_{s,r}(x^{\vec{\beta}, \vec{j}}) \in I \quad \text{for any } \vec{\beta} \in \Gamma, \vec{j} \in \vec{\mathcal{J}}. \quad (3.78)$$

Therefore, $I = \mathcal{S}$. \square

4 Algebras of Type H

In this section, we shall construct and prove a new class of generalized simple Lie algebras of Hamiltonian type.

All the notations and assumptions except (2.6) and (2.7) are the same as in Section 2. Moreover, we allow $n = 0$. Assume

$$n = 2m \quad \text{for some } m \in \mathbb{N} \quad (4.1)$$

and allow $m = 0$. Moreover, we shall use the settings in (2.5), (2.8)-(2.14). Let $m_1 \in \mathbb{N}$ such that $m_1 \leq m$. We view Γ as a \mathbb{Z} -module. Let $\phi(\cdot, \cdot) : \Gamma \times \Gamma \rightarrow \mathbb{F}$ be a skew-symmetric \mathbb{Z} -bilinear form. We shall replace the assumptions (2.6) and (2.7) as follows. First, we assume

$$\varphi_p(\Gamma) + \mathcal{J}_p \neq \{0\} \quad \text{for } p \in \overline{1, n}; \quad (4.2)$$

$$\varphi_p \neq 0 \text{ or } \varphi_{m+p} \neq 0 \quad \text{for } p \in \overline{1, m_1}; \quad (4.3)$$

$$\varphi_{m+q} = 0 \text{ if } \mathcal{J}_q = \{0\} \text{ and } \varphi_q = 0 \text{ if } \mathcal{J}_{m+q} = \{0\} \quad (4.4)$$

for $q \in \overline{m_1 + 1, m}$. Set

$$\mathcal{U} = \{p, m+p \mid p \in \overline{1, m_1}, \varphi_p \neq 0, \varphi_{m+p} \neq 0\}. \quad (4.5)$$

Furthermore, we assume

$$\left(\bigcap_{p \neq q \in \overline{1, n}} \ker \varphi_q \right) \setminus \ker \varphi_p \neq \emptyset \quad \text{if } \varphi_p \neq 0, \quad p \in \overline{m_1 + 1, m} \cup \overline{m + m_1 + 1, n}; \quad (4.6)$$

$$\text{Rad}_\phi \bigcap \left(\bigcap_{p \neq q \in \overline{1, n}} \ker_{\varphi_q} \right) \setminus \ker_{\varphi_p} \neq \emptyset \quad \text{if } \varphi_p \neq 0, \quad p \in \overline{1, m_1} \cup \overline{m+1, m+m_1}; \quad (4.7)$$

$$\{\alpha \in \Gamma \mid \phi(\alpha, \beta) = 0 \text{ for } \beta \in \bigcap_{q \in \mathcal{U}} \ker_{\varphi_q}\} \bigcap \left(\bigcap_{p=1}^n \ker_{\varphi_p} \right) = \{0\}. \quad (4.8)$$

We choose fixed elements

$$0 \neq \sigma_p = \sigma_{m+p} \in \text{Rad}_\phi \bigcap \left(\bigcap_{p, m+p \neq q \in \overline{1, n}} \ker_{\varphi_q} \right) \quad \text{for } p \in \overline{1, m_1}. \quad (4.9)$$

Set

$$\sigma = \sum_{p=1}^{m_1} \sigma_p. \quad (4.10)$$

For any $\alpha \in \Gamma$, we set

$$\mathcal{A}_\alpha = \text{span} \{x^{\alpha, \vec{j}} \mid \vec{j} \in \vec{\mathcal{J}}\}. \quad (4.11)$$

Moreover, we use notions in (3.1). Define an algebraic operation $[\cdot, \cdot]$ on \mathcal{A} by:

$$\begin{aligned} [u, v] &= \sum_{q=m_1+1}^m [\partial_q(u) \hat{\partial}_{m+q}(v) + \hat{\partial}_q(u) \partial_{\varphi_{m+q}}(v) - \hat{\partial}_{m+q}(u) \partial_q(v) - \partial_{\varphi_{m+q}}(u) \hat{\partial}_q(v)] \\ &\quad + \sum_{p=1}^{m_1} x_1^{\sigma_p} [\partial_p(u) \partial_{m+p}(v) - \partial_{m+p}(u) \partial_p(v)] + \phi(\alpha, \beta) uv \end{aligned} \quad (4.12)$$

for $u \in \mathcal{A}_\alpha$ and $v \in \mathcal{A}_\beta$ (cf. (2.2.12), (2.2.14)). It can be verified that the pair $(\mathcal{A}, [\cdot, \cdot])$ forms a Lie algebra. Obviously, 1 is a central element of \mathcal{A} . Form a quotient algebra

$$H = \mathcal{A}/\mathbb{F}, \quad (4.13)$$

whose induced Lie bracket is also denoted by $[\cdot, \cdot]$ when the context is clear. We call the Lie algebra $(H, [\cdot, \cdot])$ a *generalized Lie algebra of Hamiltonian type*.

The following fact in linear algebra will be used.

Lemma 4.1. *Let T be a linear transformation on a vector space U and let U_1 be a subspace of U such that $T(U_1) \subset U_1$. Suppose that u_1, u_2, \dots, u_n are eigenvectors of T corresponding to different eigenvalues. If $\sum_{p=1}^n u_p \in U_1$, then $u_1, u_2, \dots, u_n \in U_1$.*

Theorem 4.2. *The quotient algebra $(H, [\cdot, \cdot])$ is a simple Lie algebra if $\vec{\mathcal{J}} \neq \{\vec{0}\}$ or $\sigma = 0$. If $\vec{\mathcal{J}} = \{\vec{0}\}$ and $\sigma \neq 0$, then $H^{(1)} = [H, H]$ is a simple Lie algebra and $H = H^{(1)} \oplus (\mathbb{F}x_1^\sigma + \mathbb{F})$.*

Proof. We divide our proof as two parts.

Proof of the First Statement in the Theorem

Note that the first statement in the theorem is equivalent to that any ideal of \mathcal{A} that strictly contains \mathbb{F} is equal to \mathcal{A} . Let \mathcal{I} be an ideal of \mathcal{A} such that $\mathcal{I} \supset \mathbb{F}$ and $\mathcal{I} \neq \mathbb{F}$.

Step 1. $x_1^\alpha \in \mathcal{I}$ for some $0 \neq \alpha \in \Gamma$ or $x_2^{1[p]} \in \mathcal{I}$ for some $\mathcal{J}_p = \mathbb{N}$.

For any $\vec{j} \in \vec{\mathcal{J}}$, we define

$$|\vec{j}| = \sum_{p=1}^n j_p. \quad (4.14)$$

Set

$$\mathcal{A}_k = \text{span} \{x^{\alpha, \vec{j}} \mid (\alpha, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}, |\vec{j}| \leq k\} \quad \text{for } k \in \mathbb{N}. \quad (4.15)$$

For convenience, we let

$$\mathcal{A}_{-1} = \emptyset. \quad (4.16)$$

Moreover, we define

$$\hat{k} = \min \{k \in \mathbb{N} \mid (\mathcal{A}_k \cap \mathcal{I}) \setminus \mathbb{F} \neq \emptyset\}. \quad (4.17)$$

For any $u \in (\mathcal{A}_{\hat{k}} \cap \mathcal{I}) \setminus \mathbb{F}$, we write:

$$u = u_{ld} + \tilde{u} \quad \text{with } \tilde{u} \in \mathcal{A}_{\hat{k}-1} + \mathbb{F} \quad (4.18)$$

and

$$u_{ld} = \sum_{(0, \vec{0}) \neq (\alpha, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}, |\vec{j}| = \hat{k}} a_{\alpha, \vec{j}} x^{\alpha, \vec{j}}, \quad a_{\alpha, \vec{j}} \in \mathbb{F} \quad (4.19)$$

and define

$$\flat(u) = |\{\alpha \in \Gamma \mid a_{\alpha, \vec{j}} \neq 0 \text{ for some } \vec{j} \in \vec{\mathcal{J}}, |\vec{j}| = \hat{k}\}|. \quad (4.20)$$

Furthermore, we set

$$\flat = \min \{\flat(v) \mid v \in (\mathcal{A}_{\hat{k}} \cap \mathcal{I}) \setminus \mathbb{F}\}. \quad (4.21)$$

Let $u \in (\mathcal{A}_{\hat{k}} \cap \mathcal{I}) \setminus \mathbb{F}$ such that $\flat(u) = \flat$. Write u as in (4.18) and (4.19). We set

$$q' = m + q, \quad (m + q)' = q, \quad \epsilon_q = 1, \quad \epsilon_{m+q} = -1 \quad \text{for } q \in \overline{1, m}. \quad (4.22)$$

If $\varphi_p \neq 0$ for some $p \in \overline{m_1 + 1, m} \cup \overline{m + m_1 + 1, n}$, then $\mathcal{J}_{p'} = \mathbb{N}$ by (4.4). Thus

$$[x_2^{1[p']}, u] \equiv \sum_{(0, \vec{0}) \neq (\alpha, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}, |\vec{j}| = \hat{k}} \epsilon_{p'} a_{\alpha, \vec{j}} \varphi_p(\alpha) x^{\alpha, \vec{j}} \pmod{\mathcal{A}_{\hat{k}-1}}. \quad (4.23)$$

By the minimality of $\flat(u)$ (cf. (4.21)) and Lemma 4.1,

$$\varphi_p(\alpha) = \varphi_p(\beta) \quad \text{whenever } a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0. \quad (4.24)$$

Assume that $\varphi_p \neq 0$ and $\varphi_{p'} \neq 0$ for some $p \in \overline{1, m_1} \cup \overline{m+1, m+m_1}$. Then for any $\tau \in \text{Rad}_\phi \cap (\bigcap_{p \neq q \in \overline{1, n}} \ker_{\varphi_q})$, we have

$$\begin{aligned} & [x_1^{-\tau-2\sigma_p}, [u, x_1^\tau]] \\ \equiv & \sum_{(0, \vec{0}) \neq (\alpha, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}, |\vec{j}| = \hat{k}} \varphi_p(\tau) \varphi_{p'}(\alpha) (\varphi_p(\tau) \varphi_{p'}(\alpha - \sigma_p) + 2\varphi_p(\sigma_p) \varphi_{p'}(\alpha + \sigma_p) \\ & - 2\varphi_p(\alpha + \sigma_p) \varphi_{p'}(\sigma_p)) a_{\alpha, \vec{j}} x^{\alpha, \vec{j}} \pmod{\mathcal{A}_{\hat{k}-1}}. \end{aligned} \quad (4.25)$$

Since $\varphi_p(\tau)$ takes an infinite number of elements in \mathbb{F} if τ varies in $\text{Rad}_\phi \cap (\bigcap_{p \neq q \in \overline{1, n}} \ker_{\varphi_q})$ by (4.7), the coefficients of $\varphi_p(\tau)^2$ in (4.25) show

$$\varphi_{p'}(\alpha) \varphi_{p'}(\alpha - \sigma_p) = \varphi_{p'}(\beta) \varphi_{p'}(\beta - \sigma_p) \quad \text{whenever } a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0 \quad (4.26)$$

by the minimality of $\mathfrak{b}(u)$ (cf. (4.21)) and Lemma 4.1. Moreover, (4.26) is equivalent to

$$\varphi_{p'}(\alpha) = \varphi_{p'}(\beta) \quad \text{or} \quad \varphi_{p'}(\alpha + \beta - \sigma_p) = 0 \quad \text{whenever } a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0. \quad (4.27)$$

Assume that there exist $\alpha, \beta \in \Gamma$ such that $\varphi_{p'}(\alpha) \neq \varphi_{p'}(\beta)$, $\varphi_{p'}(\alpha + \beta - \sigma_p) = 0$ and $a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0$. We may assume that $\varphi_{p'}(\alpha) \neq 0$. Since $\varphi_p \neq 0$ and $\varphi_{p'} \neq 0$, we can choose

$$\tau \in \text{Rad}_\phi \cap \left(\bigcap_{p \neq q \in \overline{1, n}} \ker_{\varphi_q} \right) \setminus \ker_{\varphi_p}, \quad \tau' \in \text{Rad}_\phi \cap \left(\bigcap_{p' \neq q \in \overline{1, n}} \ker_{\varphi_q} \right) \setminus \ker_{p'}, \quad (4.28)$$

Such that

$$\varphi_{p'}(\tau' + \alpha) \neq 0, \quad \varphi_p(\tau - \sigma_p) \varphi_{p'}(\alpha) \neq \varphi_p(\alpha) \varphi_{p'}(\tau' - \sigma_p) \quad (4.29)$$

by (4.7). We have

$$\begin{aligned} & [x_1^{\tau+\tau'-\sigma_p}, u] \equiv \sum_{(0, \vec{0}) \neq (\gamma, \vec{l}) \in \Gamma \times \vec{\mathcal{J}}, |\vec{l}| = \hat{k}} \epsilon_p a_{\gamma, \vec{l}} (\varphi_p(\tau - \sigma_p) \varphi_{p'}(\gamma) \\ & - \varphi_p(\gamma) \varphi_{p'}(\tau' - \sigma_p)) x^{\gamma+\tau+\tau', \vec{l}} \pmod{\mathcal{A}_{\hat{k}-1}}. \end{aligned} \quad (4.30)$$

Since $\mathfrak{b}(u)$ is minimum, $\alpha + \tau + \tau' \neq 0$ due to $\varphi_{p'}(\alpha + \tau + \tau') \neq 0$ and

$$\epsilon_p a_{\alpha, \vec{j}} (\varphi_p(\tau - \sigma_p) \varphi_{p'}(\alpha) - \varphi_p(\alpha) \varphi_{p'}(\tau' - \sigma_p)) \neq 0, \quad (4.31)$$

we have

$$\epsilon_p a_{\beta, \vec{j}'} (\varphi_p(\tau - \sigma_p) \varphi_{p'}(\beta) - \varphi_p(\beta) \varphi_{p'}(\tau' - \sigma_p)) \neq 0 \quad (4.32)$$

by Lemma 4.1. But

$$\varphi_{p'}(\alpha + \tau + \tau') \neq \varphi_{p'}(\beta + \tau + \tau') \quad (4.33)$$

and

$$\varphi_{p'}((\alpha + \tau + \tau') + (\beta + \tau + \tau') - \sigma_p) = 2\varphi_{p'}(\tau') \neq 0, \quad (4.34)$$

which contradicts (4.27) with u replaced by $[x_1^{\tau+\tau'-\sigma_p}, u]$. Thus the first equation in (4.27) holds.

Assume $\varphi_p = 0$ with $p \in \overline{1, m_1} \cup \overline{m+1, m+m_1}$. Then $\mathcal{J}_p = \mathbb{N}$ by (4.2). If $j_p \neq 0$ for some $a_{\alpha, \vec{j}} \neq 0$, then $[x_1^{l\sigma_p}, u] \in \mathcal{I} \cap \mathcal{A}_{\hat{k}-1} \setminus \mathbb{F}$ for some $0 < l \in \mathbb{N}$ such that $(l+1)\sigma_p + \alpha \neq 0$ by (4.9), which contradicts (4.17). Assume that $j_p = 0$ whenever $a_{\alpha, \vec{j}} \neq 0$. In this case, we get

$$[x_2^{-\sigma_p, 1_{[p]}}, u] \equiv \sum_{(0, \vec{0}) \neq (\alpha, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}, |\vec{j}| = \hat{k}} a_{\alpha, \vec{j}} \epsilon_p \varphi_{p'}(\alpha) x^{\alpha, \vec{j}} \pmod{\mathcal{A}_{\hat{k}-1}}, \quad (4.35)$$

which implies the first equation in (4.27). Therefore, we have proved that (4.24) for any $p \in \overline{1, n}$.

If $a_{\beta, \vec{j}'} \neq 0$, then

$$\mathcal{I} \ni [x_1^\beta, u] \equiv \sum_{(0, \vec{0}) \neq (\alpha, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}, |\vec{j}| = \hat{k}} a_{\alpha, \vec{j}} \phi(\beta, \alpha) x^{\alpha + \beta, \vec{j}} \pmod{\mathcal{A}_{\hat{k}-1}} \quad (4.36)$$

by (4.12) and (4.24). Since $\mathfrak{b}(u)$ is minimum and $\phi(\beta, \beta) = 0$, we have

$$\phi(\alpha, \beta) = 0 \quad \text{whenever} \quad a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0. \quad (4.37)$$

Now for any $\gamma \in \bigcap_{q \in U} \ker \varphi_q$, we have

$$\mathcal{I} \ni [[x_1^{-\gamma}, [x_1^\gamma, u]] \equiv \sum_{(0, \vec{0}) \neq (\alpha, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}, |\vec{j}| = \hat{k}} -a_{\alpha, \vec{j}} \phi(\gamma, \alpha)^2 x^{\alpha, \vec{j}} \pmod{\mathcal{A}_{\hat{k}-1}}. \quad (4.38)$$

Since $\mathfrak{b}(u)$ is minimum, by Lemma 4.1, we get

$$\phi(\gamma, \alpha)^2 = \phi(\gamma, \beta)^2 \quad \text{whenever} \quad a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0, \quad (4.39)$$

which is equivalent to

$$\phi(\gamma, \alpha) = \pm \phi(\gamma, \beta) \quad \text{whenever} \quad a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0. \quad (4.40)$$

Assume that there exist $\alpha, \beta \in \Gamma$ such that $\phi(\gamma, \alpha) = -\phi(\gamma, \beta) \neq 0$ and $a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0$. Then

$$\mathcal{I} \ni [x_1^\gamma, u] \equiv \sum_{(0, \vec{0}) \neq (\alpha, \vec{j}) \in \Gamma \times \vec{\mathcal{J}}, |\vec{j}| = \hat{k}} a_{\alpha, \vec{j}} \phi(\gamma, \alpha) x^{\alpha + \gamma, \vec{j}} \pmod{\mathcal{A}_{\hat{k}-1}}. \quad (4.41)$$

Thus $\mathfrak{b}([x_1^\gamma, u]) = \mathfrak{b}(u)$ and

$$a_{\alpha, \vec{j}} \phi(\gamma, \alpha) a_{\beta, \vec{j}} \phi(\gamma, \beta) \neq 0. \quad (4.42)$$

But

$$\phi(\beta + \gamma, \alpha + \gamma) = \phi(\gamma, \alpha) + \phi(\beta, \gamma) = 2\phi(\gamma, \alpha) \neq 0 \quad (4.43)$$

by (4.37) and the skew-symmetry of ϕ . Equation (4.43) contradicts (4.37) if we replace u by $[x_1^\gamma, u]$. Hence we have

$$\phi(\gamma, \alpha) = \phi(\gamma, \beta) \quad \text{whenever } a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0 \quad (4.44)$$

for any $\gamma \in \bigcap_{q \in \mathcal{U}} \ker \varphi_q$. By (4.8), (4.24) and (4.44), we obtain

$$\alpha = \beta \quad \text{whenever } a_{\alpha, \vec{j}} a_{\beta, \vec{j}'} \neq 0. \quad (4.45)$$

Let $a_{\alpha, \vec{j}} \neq 0$ be fixed. If $\hat{k} = 0$, then we have $x_1^\alpha \in \mathcal{I}$. Assume that $\hat{k} > 0$.

Case 1. $\alpha = 0$ and $\hat{k} = 1$.

In this case,

$$u = \sum_{p=1}^n a_p x_2^{1_{[p]}} + \sum_{\beta \in \Gamma} b_\beta x_1^\beta. \quad (4.46)$$

For convenience of stating things, we set

$$\sigma_p = \sigma_{m+p} = 0 \quad \text{for } p \in \overline{m_1 + 1, m}. \quad (4.47)$$

If $\varphi_p \neq 0$ with $p \in \overline{1, n}$, we choose $-\sigma_p \neq \tau \in (\bigcap_{p \neq q \in \overline{1, n}} \ker \varphi_q) \setminus \ker \varphi_p$ by (4.6), (4.7) and have

$$[x_1^\tau, u] = \epsilon_p \varphi_p(\tau) a_{p'} x_1^{\tau + \sigma_p} + \sum_{\beta \in \Gamma} b_\beta (\phi(\tau, \beta) x_1^{\beta + \tau} + \epsilon_p \varphi_p(\tau) \varphi_{p'}(\beta) x_1^{\beta + \tau + \sigma_p}) \in \mathcal{I}. \quad (4.48)$$

If $a_{p'} \neq 0$, then $[x_1^\tau, u] \notin \mathbb{F}$, which contradicts the assumption $\hat{k} = 1$.

If $\varphi_p = 0$, then $\mathcal{J}_p = \mathbb{N}$. We have

$$[x_2^{1_{[p]}}, u] = \epsilon_p a_{p'} x_1^{\sigma_p} + \sum_{\beta \in \Gamma} \epsilon_p b_\beta \varphi_{p'}(\beta) x_1^{\sigma_p + \beta} \in \mathcal{I}. \quad (4.49)$$

If $a_{p'} \neq 0$ and $\epsilon_p b_\beta \varphi_{p'}(\beta) \neq 0$ for some $\beta \in \Gamma$, we get a contradiction to the assumption $\hat{k} = 1$. We assume $a_{p'} \neq 0$ for some $p' \in \overline{1, n}$, then we have $\varphi_p = 0$ by (4.48) and

$$\epsilon_p b_\beta \varphi_{p'}(\beta) = 0 \quad \text{for } \beta \in \Gamma \quad (4.50)$$

by (4.49). Note $\mathcal{J}_p = \mathbb{N}$ in this case. So

$$[x_2^{2_{[p]}}, u] = 2\epsilon_p a_{p'} x_2^{1_{[p]}} \in \mathcal{I}. \quad (4.51)$$

Thus the conclusion of Step 1 holds.

Case 2. $\alpha \neq 0$ or $\hat{k} > 1$. $j'_p \neq 0$ for some $a_{\alpha, \vec{j}} \neq 0$ with $p \in \overline{1, n}$ and $\varphi_p(\alpha) = 0$.

If $p \in \overline{1, m_1} \cap \overline{m+1, m+m_1}$ and $\varphi_{p'} \neq 0$, we choose $\tau' \in \text{Rad}_\phi \cap (\bigcap_{p' \neq q \in \overline{1, n}} \ker_{\varphi_q}) \setminus \ker_{\varphi_{p'}}$ such that $\varphi_{p'}(\alpha + \tau' + \sigma_p) \neq 0$ by (4.7). Then

$$[u, x_1^{\tau'}] \in (\mathcal{I} \cap \mathcal{A}_{\hat{k}-1}) \setminus \mathbb{F}, \quad (4.52)$$

which contradicts (4.17).

If $p \in \overline{m_1+1, m} \cap \overline{m+m_1+1, n}$, $\varphi_{p'} \neq 0$, we let $\tau' \in (\bigcap_{p' \neq q \in \overline{1, n}} \ker_{\varphi_q}) \setminus \ker_{\varphi_{p'}}$ such that $\varphi_{p'}(\alpha + \tau') \neq 0$ by (4.6). If $\phi(\tau', \alpha) = 0$, then we have (4.52). Otherwise,

$$\begin{aligned} v &= [x_1^{\alpha+\tau'}, [x_1^{\tau'}, u]] \\ &\equiv \sum_{\vec{j}'' \in \vec{\mathcal{J}}, |\vec{j}''|=\hat{k}} \phi(\tau', \alpha) a_{\alpha, \vec{j}''} (\varphi_{p'}(\tau') + \varphi_{p'}(\alpha)) j_p'' x^{\alpha, \vec{j}''-1_{[p]}} + w \pmod{\mathcal{A}_{\hat{k}-2}}, \end{aligned} \quad (4.53)$$

where

$$w = \sum_{(\beta, \vec{j}'') \in \Gamma \times \vec{\mathcal{J}}, |\vec{j}''|=\hat{k}-1} b_{\beta, \vec{j}''} x^{\beta, \vec{j}''} \quad (4.54)$$

such that $b_{\beta, \vec{j}''} \in \mathbb{F}$ are independent of $\varphi_{p'}(\tau')$ and the number of nonzero $b_{\beta, \vec{j}''}$ is bounded with respect to $\varphi_{p'}(\tau')$. Since $|\varphi_{p'}(\mathbb{Z}\tau')| = \infty$, $\varphi_{p'}(\tau')$ takes an infinite number of elements in \mathbb{F} when τ' varies in $\tau' \in (\bigcap_{p' \neq q \in \overline{1, n}} \ker_{\varphi_q}) \setminus \ker_{\varphi_{p'}}$ such that $\phi(\tau', \alpha) \neq 0$. Hence we can choose τ' such that $v \in \mathcal{A}_{\hat{k}-1} \setminus \mathbb{F}$, which contradicts (4.17).

Assume $\varphi_{p'} = 0$. We have $\mathcal{J}_{p'} = \mathbb{N}$ by (4.2). Then

$$[u, x_2^{1_{[p']}}] \in (\mathcal{I} \cap \mathcal{A}_{\hat{k}-1}) \setminus \mathbb{F}, \quad (4.55)$$

which contradicts (4.17).

Case 3. $j'_p \neq 0$ for some $a_{\alpha, \vec{j}'} \neq 0$ with $p \in \overline{m_1+1, m} \cup \overline{m+m_1+1, n}$ and $\varphi_p(\alpha) \neq 0$.

Note that $[x_1^{-\alpha}, u] \in \mathcal{I} \cap \mathcal{A}_{\hat{k}-1}$. By (4.17), we have

$$[x_1^{-\alpha}, u] = \lambda \in \mathbb{F}. \quad (4.56)$$

Thus we get

$$\begin{aligned} \mathcal{I} \ni [u, x_1^{-\alpha} x_2^{1_{[p']}}] &= [u, x_1^{-\alpha}] x_2^{1_{[p']}} + x_1^{-\alpha} [u, x_2^{1_{[p']}}] \\ &\equiv \lambda x_2^{1_{[p']}} + \sum_{\vec{j}'' \in \vec{\mathcal{J}}, |\vec{j}''|=\hat{k}} \varphi_p(\alpha) \epsilon_p a_{\alpha, \vec{j}''} x^{0, \vec{j}''} \pmod{\mathcal{A}_{\hat{k}-1}}. \end{aligned} \quad (4.57)$$

Replacing u by $[u, x_1^{-\alpha} x_2^{1_{[p']}}]$, we go back to Case 2 if $\hat{k} > 1$ and to Case 1 if $\hat{k} = 1$ because $j'_p \neq 0$.

Case 4. There exists $p \in \overline{1, m_1} \cup \overline{m+1, m+m_1}$ such that $\varphi_p(\alpha) \neq 0$, $\varphi_{p'} \neq 0$ and $j_p > 0$.

We can choose $\tau' \in \text{Rad}_\phi \cap (\bigcap_{p' \neq q \in \overline{1, n}} \ker_{\varphi_q}) \setminus \ker_{\varphi_{p'}}$ such that $\varphi_{p'}(\tau' + \sigma_p)\varphi_p(\alpha) \neq \varphi_p(\sigma_p)\varphi_{p'}(\alpha)$ by (4.7). Note that

$$\begin{aligned} \mathcal{I} \ni [u, x_1^{-\alpha-\tau'-\sigma_p}] &\equiv \epsilon_{p'}(\varphi_{p'}(\tau' + \sigma_p)\varphi_p(\alpha) - \varphi_p(\sigma_p)\varphi_{p'}(\alpha)) \\ &\quad \sum_{\vec{j}' \in \vec{\mathcal{J}}, |\vec{j}'| = \hat{k}} a_{\alpha, \vec{j}'} x^{\tau', \vec{j}'} \pmod{\mathcal{A}_{\hat{k}-1}}. \end{aligned} \quad (4.58)$$

Since $\varphi_p(\tau') = 0$ and $j_p > 0$, we go back to Case 2 with u replaced by $[u, x_1^{-\alpha-\tau'-\sigma_p}]$.

Case 5. There exists $p \in \overline{1, m_1} \cup \overline{m+1, m+m_1}$ such that $\varphi_p(\alpha) \neq 0$, $\varphi_{p'} = 0$, $j_p > 0$ and $j_{p'} = 0$ for all $a_{\alpha, \vec{j}'} \neq 0$.

By (4.2), we have $\mathcal{J}_{p'} = \mathbb{N}$. Note that (4.56) holds. As (4.57), we have

$$\begin{aligned} \mathcal{I} \ni [u, x_1^{-\alpha} x^{-\sigma_p, 1_{[p']}}] &= [u, x_1^{-\alpha}] x^{-\sigma_p, 1_{[p']}} + x_1^{-\alpha} [u, x^{-\sigma_p, 1_{[p']}}] \\ &\equiv \lambda x^{-\sigma_p, 1_{[p']}} + \sum_{\vec{j}'' \in \vec{\mathcal{J}}, |\vec{j}''| = \hat{k}} \varphi_p(\alpha) \epsilon_p a_{\alpha, \vec{j}''} x^{0, \vec{j}''} \pmod{\mathcal{A}_{\hat{k}-1}}. \end{aligned} \quad (4.59)$$

If $\hat{k} > 1$ or $\lambda = 0$, we go back to Case 2 if we replace u by $[u, x_1^{-\alpha} x^{-\sigma_p, 1_{[p']}}]$. Assume that $\lambda \neq 0$ and $\hat{k}=1$. Then we have

$$\mathcal{I} \ni [[u, x_1^{-\alpha} x^{-\sigma_p, 1_{[p']}}], x_2^{1_{[p']}}] \equiv -\epsilon_p \lambda \varphi_p(\sigma_p) x_2^{1_{[p']}} \pmod{\mathcal{A}_0}. \quad (4.60)$$

Replacing u by $[[u, x_1^{-\alpha} x^{-\sigma_p, 1_{[p']}}], x_2^{1_{[p']}}]$, we go back to Case 1.

This completes the proof of the conclusion in Step 1.

Step 2. The conclusion of Step 1 implies $\mathcal{I} = \mathcal{A}$.

Case 1. $x_2^{1_{[p]}} \in \mathcal{I}$ for some $p \in \overline{1, n}$.

In this case, $\mathcal{J}_p = \mathbb{N}$. For any $(\beta, \vec{j}') \in \Gamma \times \vec{\mathcal{J}}$, we have

$$[x_2^{1_{[p]}}, x^{\beta, \vec{j}'}] = \epsilon_p(\varphi_{p'}(\beta) x^{\beta+\sigma_p, \vec{j}'} + j'_p x^{\beta+\sigma_p, \vec{j}'-1_{[p]}}) \in \mathcal{I}. \quad (4.61)$$

If $\mathcal{J}_{p'} = \mathbb{N}$, then by (4.61) and induction on $j'_{p'}$, we can prove $\mathcal{I} = \mathcal{A}$. Assume $\mathcal{J}_{p'} = \{0\}$. By (4.2), $\varphi_{p'} \neq 0$. We choose $\tau' \in (\bigcap_{p' \neq q \in \overline{1, n}} \ker_{\varphi_q}) \setminus \ker_{\varphi_{p'}}$ such that $\varphi_{p'}(\tau' - \sigma_p) \neq 0$ by (4.6) and (4.7). Moreover, we can assume $\tau' \in \text{Rad}_\phi$ if $p' \in \overline{1, m_1} \cup \overline{m+1, m+m_1}$ by (4.7). We have

$$[x_2^{1_{[p]}}, x_1^{\tau'-\sigma_p}] = \epsilon_p \varphi_{p'}(\tau' - \sigma_p) x_1^{\tau'} \in \mathcal{I}. \quad (4.62)$$

So $x_1^{\tau'} \in \mathcal{I}$. Moreover, for any $(\beta, \vec{j}') \in \Gamma \times \vec{\mathcal{J}}$,

$$[x_1^{\tau'}, x^{\beta, \vec{j}'}] = \epsilon_{p'} \varphi_{p'}(\tau') (\varphi_p(\beta) x^{\beta+\tau'+\sigma_p, \vec{j}'} + j'_p x^{\beta+\sigma_p+\tau', \vec{j}'-1_{[p]}}) \in \mathcal{I} \quad (4.63)$$

if $p \in \overline{1, m_1} \cup \overline{m+1, m+m_1}$ and

$$[x_1^{\tau'}, x^{\beta, \vec{j}'}] = \phi(\tau', \beta) x^{\beta+\tau', \vec{j}'} + \epsilon_{p'} \varphi_{p'}(\tau') j'_p x^{\beta+\tau', \vec{j}'-1_{[p]}} \in \mathcal{I} \quad (4.64)$$

if $p \in \overline{m_1+1, m} \cup \overline{m+m_1+1, n}$. Thus by (4.63), (4.64) and induction on j'_p , we can prove $\mathcal{I} = \mathcal{A}$.

Case 2. $x_1^\alpha \in \mathcal{I}$ for some $0 \neq \alpha \in \Gamma$ and $\varphi_p(\alpha) \neq 0$ for some $p \in \overline{1, n}$.

Assume $p \in \overline{1, m_1} \cup \overline{m+1, m+m_1}$ and $\varphi_{p'} = 0$ or $p \in \overline{m_1+1, m} \cup \overline{m+m_1+1, n}$. We have $\mathcal{J}_{p'} = \mathbb{N}$ by (4.2) and (4.4). Moreover,

$$[x_1^\alpha, x^{-\alpha-\sigma_p, 2_{[p']}}] = 2\epsilon_p \varphi_p(\alpha) x_2^{1_{[p']}} \in \mathcal{I}. \quad (4.65)$$

So $x_2^{1_{[p']}} \in \mathcal{I}$. Hence $\mathcal{I} = \mathcal{A}$ by Case 1. Next we consider $p \in \overline{1, m_1} \cup \overline{m+1, m+m_1}$ and $\varphi_{p'} \neq 0$. By (4.58) with $u = x_1^\alpha$, we obtain $x_1^{\tau'} \in \mathcal{I}$ for some $\tau' \in \text{Rad}_\phi \cap (\bigcap_{p' \neq q \in \overline{1, n}} \ker \varphi_q) \setminus \ker \varphi_{p'}$. Furthermore, by (4.63), we have $\mathcal{I} = \mathcal{A}$ if $\mathcal{J}_p = \mathbb{N}$ and otherwise,

$$x^{\beta, \vec{j}'} \in \mathcal{I} \quad \text{for any } (\beta, \vec{j}') \in \Gamma \times \vec{\mathcal{J}}, \varphi_p(\beta) \neq \varphi_p(\sigma_p). \quad (4.66)$$

Choose any $\tau \in (\bigcap_{p \neq q \in \overline{1, n}} \ker \varphi_q) \setminus \ker \varphi_p$ such that $\varphi_p(\tau) \neq \varphi_p(\sigma_p)$. We have $x_1^{\tau+\tau'} \in \mathcal{I}$ by (4.66) and $\varphi_{p'}(\tau + \tau') = \varphi_{p'}(\tau') \neq 0$. Exchanging positions of p and p' , we have the similar conclusion. If $\varphi_q \neq 0$ for some $q \in \overline{1, n} \setminus \{p, p'\}$, then we choose any $\tau_q \in (\bigcap_{q \neq r \in \overline{1, n}} \ker \varphi_r) \setminus \ker \varphi_q$ and have $x_1^{\tau+\tau_q} \in \mathcal{I}$, $\varphi_q(\tau + \tau_q) = \varphi_q(\tau_q) \neq 0$. Thus we have the same conclusion for (q, q') as that for (p, p') . Assume that $\varphi_q = \varphi_{q'} = 0$ for some $q \in \overline{1, n} \setminus \{p, p'\}$, then $\mathcal{J}_q = \mathcal{J}_{q'} = \mathbb{N}$. Since $x^{\tau, 1_{[q]}} \in \mathcal{I}$, we have

$$[x^{\tau, 1_{[q]}}, x^{-\tau, 2_{[q']}}] = 2\epsilon_q x_2^{1_{[q']}} \in \mathcal{I}, \quad (4.67)$$

by which and Case 1, we get $\mathcal{I} = \mathcal{A}$. In summary, we have $\mathcal{I} = \mathcal{A}$ if $\vec{\mathcal{J}} \neq \{\vec{0}\}$ and

$$x_1^\beta \in \mathcal{I} \quad \text{for } \beta \in \Gamma \text{ such that } \beta - \sigma \notin \bigcap_{r=1}^n \ker \varphi_r \quad (4.68)$$

(cf. (4.10)) if $\vec{\mathcal{J}} = \{\vec{0}\}$, where $m_1 = m$ by (4.2) and (4.4).

Assume $\vec{\mathcal{J}} = \{\vec{0}\}$. Let $\beta \in \Gamma$ such that $0 \neq \beta - \sigma \in \bigcap_{r=1}^n \ker \varphi_r$. By (4.8), there exists $\gamma \in \bigcap_{r=1}^n \ker \varphi_r$ such that $\phi(\gamma, \beta) = \phi(\gamma, \beta - \sigma) \neq 0$, where we have used the fact that $\sigma \in \text{Rad}_\phi$ by (4.9) and (4.10). Moreover, we choose $\tau \in \text{Rad}_\phi \cap (\bigcap_{p \neq q \in \overline{1, n}} \ker \varphi_q) \setminus \ker \varphi_p$. Since $\varphi_p(\beta - \tau - \gamma) = \varphi_p(\sigma_p) - \varphi(\tau) \neq \varphi_p(\sigma_p)$, we have $x_1^{\beta-\tau-\gamma} \in \mathcal{I}$ by (4.66). Note that

$$[x_1^{\gamma+\tau}, x_1^{\beta-\gamma-\tau}] = \phi(\gamma, \beta) x_1^\beta + \epsilon_p \varphi_p(\tau) \varphi_{p'}(\sigma_p) x^{\beta+\sigma_p} \in \mathcal{I}, \quad (4.69)$$

where we have used the fact that $\varphi_{p'}(\beta) = \varphi_{p'}(\sigma_p)$ and $\varphi_{p'}(\tau) = 0$. Moreover, we have $\epsilon_p \varphi_p(\tau) \varphi_{p'}(\sigma_p) x^{\beta + \sigma_p} \in \mathcal{I}$ by (4.68). Hence $x_1^\beta \in \mathcal{I}$. Therefore, we obtain

$$x_1^\beta \in \mathcal{I} \quad \text{for } \sigma \neq \beta \in \Gamma. \quad (4.70)$$

Case 3. $x_1^\alpha \in \mathcal{I}$ for some $0 \neq \alpha \in \Gamma$ and $\varphi_p(\alpha) = 0$ for any $p \in \overline{1, n}$.

By (4.8), there exists $\beta \in \Gamma$ such that $\phi(\beta, \alpha) \neq 0$. If $\varphi_p \neq 0$ for some $p \in \overline{1, n}$, we can choose $\tau \in (\bigcap_{p \neq q \in \overline{1, n}} \ker \varphi_q) \setminus \ker \varphi_p$ such that $\varphi_p(\tau + \alpha + \beta) \neq 0$ and $\phi(\beta + \tau, \alpha) \neq 0$. Since

$$\phi(\beta + \tau, \alpha)^{-1} [x_1^{\tau + \beta}, x_1^\alpha] = x_1^{\tau + \alpha + \beta} \in \mathcal{I}, \quad (4.71)$$

we go back to Case 2. Assume that $\varphi_p = 0$ for any $p \in \overline{1, n}$ and $n > 0$. By (4.2), we have $\mathcal{J}_1 = \mathbb{N}$. Moreover,

$$[[x^{\beta, 1[1]}, x_1^\alpha], x^{-\alpha - \beta, 2[m+1]}] = 2\phi(\beta, \alpha) x_2^{1[m+1]} \in \mathcal{I}, \quad (4.72)$$

by which and Case 1, $\mathcal{I} = \mathcal{A}$. Now we assume $n = 0$. In this situation, (4.8) becomes $\text{Rad}_\phi = \{0\}$. Note

$$[x_1^{\beta - \alpha}, x_1^\alpha] = \phi(\beta, \alpha) x_1^\beta \in \mathcal{I} \quad \text{for any } \beta \in \Gamma. \quad (4.73)$$

This shows that

$$\begin{aligned} x_1^\gamma \in \mathcal{I} \text{ for } \gamma \in \Gamma, \phi(\gamma, \alpha) \neq 0 \text{ or there exists} \\ \beta \in \Gamma \text{ such that } \phi(\gamma, \beta)\phi(\beta, \alpha) \neq 0. \end{aligned} \quad (4.74)$$

If (4.74) does not imply $\mathcal{I} = \mathcal{A}$, then there exists $0 \neq \gamma \in \Gamma$ such that $\phi(\gamma, \alpha) = 0$ and

$$\phi(\gamma, \beta)\phi(\beta, \alpha) = 0 \quad \text{for any } \beta \in \Gamma. \quad (4.75)$$

Since $\text{Rad}_\phi = \{0\}$, there exists $\alpha_0, \gamma_0 \in \Gamma$ such that $\phi(\alpha_0, \alpha) \neq 0$ and $\phi(\gamma, \gamma_0) \neq 0$. By (4.75), $\phi(\gamma_0, \alpha) = \phi(\gamma, \alpha_0) = 0$. However,

$$\phi(\gamma, \alpha_0 + \gamma_0)\phi(\alpha_0 + \gamma_0, \alpha) = \phi(\gamma, \gamma_0)\phi(\alpha_0, \alpha) \neq 0, \quad (4.76)$$

which contradicts (4.75) with $\beta = \alpha_0 + \gamma_0$. Thus $\mathcal{I} = \mathcal{A}$ if $n = 0$. This completes the proof of the first statement in the theorem.

Proof of the Second Statement in the Theorem

Now $\vec{\mathcal{J}} = \{\vec{0}\}$ and $\sigma \neq 0$. By (4.4), we have $m_1 = m$. Set

$$\hat{H} = \text{span}\{x_1^\alpha \mid \sigma \neq \alpha \in \Gamma\}. \quad (4.77)$$

For any $\alpha, \beta \in \Gamma$, we have:

$$[x_1^\alpha, x_1^\beta] = \phi(\alpha, \beta)x_1^{\alpha+\beta} + \sum_{p=1}^m (\varphi_p(\alpha)\varphi_{m+p}(\beta) - \varphi_{m+p}(\alpha)\varphi_p(\beta))x_1^{\alpha+\beta+\sigma_p}. \quad (4.78)$$

If $\alpha + \beta = \sigma$, then $\phi(\alpha, \beta) = \phi(\alpha, \sigma - \alpha) = 0$ by (4.9) and (4.10). For $p \in \overline{1, m}$, if $\alpha + \beta + \sigma_p = \sigma$, then $\beta = \sum_{p \neq q \in \overline{1, m}} \sigma_q - \alpha$. So $\varphi_p(\beta) = -\varphi_p(\alpha)$ and $\varphi_{m+p}(\beta) = -\varphi_{m+p}(\alpha)$, which implies $\varphi_p(\alpha)\varphi_{m+p}(\beta) - \varphi_{m+p}(\alpha)\varphi_p(\beta) = 0$. Thus we have $[\mathcal{A}, \mathcal{A}] \subset \hat{H}$. On the other hand, $[\mathcal{A}, \mathcal{A}] \supset \hat{H}$ by (4.7), (4.8), (4.63), (4.69) and (4.73)-(4.76). Hence,

$$\hat{H} = [\mathcal{A}, \mathcal{A}]. \quad (4.79)$$

Replacing \mathcal{A} by \hat{H} in the proof of the first statement, we obtain (4.70), which implies that $H^{(1)} = \hat{H}/\mathbb{F}$ is simple. \square

Remark 4.3. Some special cases of the Lie algebra H with $m_1 = m = 1$ and $\phi = 0$ were studied in [X3].

Example. Let $k \in \mathbb{N}$. Take any nondegenerate skew-symmetric bilinear form ϕ' on \mathbb{F}^k , where we treat $\mathbb{F}^0 = \{0\}$ with $\phi' = 0$ for convenience. Suppose that we have picked (2.5) so that

$$\mathcal{J}_q = \mathbb{N} \text{ or } \mathcal{J}_{q'} = \mathbb{N} \quad \text{for } q \in \overline{m_1 + 1, m} \cup \overline{m + m_1 + 1, n} \quad (4.80)$$

(cf. (4.22)). Let k_1 be the number of $\mathcal{J}_p = \{0\}$ with $p \in \overline{1, m_1} \cup \overline{m + 1, m + m_1}$ and let k_2 be the number of $\mathcal{J}_q = \mathbb{N}$ with $q \in \overline{m_1 + 1, m} \cup \overline{m + m_1 + 1, n}$. Set $s = k_1 + n - 2m_1 - k_2$. Pick an integer ℓ such that $s \leq \ell \leq 2m_1 + k_2$. We define

$$\zeta_p(\alpha_1, \dots, \alpha_\ell) = \alpha_p \quad \text{for } p \in \overline{1, \ell}, (\alpha_1, \alpha_2, \dots, \alpha_\ell) \in \mathbb{F}^\ell. \quad (4.81)$$

Moreover, we extend ϕ' and ζ_p to $\mathbb{F}^{k+\ell}$ by

$$\phi'((\vec{\alpha}_1, \vec{\alpha}_2), (\vec{\beta}_1, \vec{\beta}_2)) = \phi'(\vec{\alpha}_1, \vec{\beta}_1), \quad \zeta_p(\vec{\alpha}_1, \vec{\alpha}_2) = \zeta_p(\vec{\alpha}_2) \quad (4.82)$$

for $\vec{\alpha}_1, \vec{\beta}_1 \in \mathbb{F}^k$, $\vec{\alpha}_2, \vec{\beta}_2 \in \mathbb{F}^\ell$ and $p \in \overline{1, \ell}$. Furthermore, we define

$$\zeta_q \equiv 0 \quad \text{for } q \in \overline{\ell + 1, n}. \quad (4.83)$$

Take Γ to be an additive subgroup of $\mathbb{F}^{k+\ell}$ containing $\mathbb{Z}^{k+\ell}$. Take any permutation ι on $\overline{1, n}$ such that

$$\iota(p) \leq \ell \text{ if } \mathcal{J}_p = \{0\} \text{ with } p \in \overline{1, m_1} \cup \overline{m + 1, m + m_1}, \quad (4.84)$$

$$\iota(q) > \ell, \iota_{q'} \leq \ell \text{ if } \mathcal{J}_{q'} = \{0\} \text{ with } q \in \overline{m_1 + 1, m} \cup \overline{m + m_1 + 1, n} \quad (4.85)$$

(cf. (4.22)). We let

$$\phi = \phi'(|_\Gamma, |_\Gamma), \quad \varphi_p = \zeta_{\iota(p)}|_\Gamma \quad \text{for } p \in \overline{1, n}. \quad (4.86)$$

Then (4.2)-(4.4) and (4.6)-(4.8) hold.

In particular, we can take $k = 0$, $m_1 = m$, (2.34) and (2.35).

5 Algebras of Type K

In this section, we shall construct and prove a new class of generalized simple Lie algebras of Contact type. We shall use the notions in Section 2. All the notations and assumptions except (2.6) are the same as in Section 2.

Now we assume

$$n = 2m + 1 \text{ for some } 0 < m \in \mathbb{N}; \quad \varphi_p \not\equiv 0 \text{ or } \mathcal{J}_p = \mathbb{N} \text{ for } p \in \overline{1, 2m + 1} \quad (5.1)$$

and

$$\Gamma = \Gamma_1 + \Gamma_2, \quad \Gamma_1 = \ker_{\varphi_n}, \quad \Gamma_2 = \bigcap_{p=1}^{2m} \ker_{\varphi_p}. \quad (5.2)$$

By (2.7), the sum $\Gamma_1 + \Gamma_2$ is a direct sum, $\varphi_n : \Gamma_2 \rightarrow \varphi_n(\Gamma_2) = \varphi_n(\Gamma)$ is an additive group homomorphism and

$$\bigcap_{p=1}^{2m} \ker_{\varphi_p|_{\Gamma_1}} = \{0\}. \quad (5.3)$$

Let

$$\mathcal{U}_1 = \{p \in \overline{1, 2m} \mid \varphi_p \not\equiv 0\}, \quad \mathcal{U}_2 = \overline{1, 2m} \setminus \mathcal{U}_1. \quad (5.4)$$

Condition (2.6) is replaced by

$$\bigcap_{q \neq p \in \overline{1, 2m}} \ker_{\varphi_p|_{\Gamma_1}} \setminus \ker_{\varphi_q|_{\Gamma_1}} \neq \emptyset \quad \text{for } q \in \mathcal{U}_1. \quad (5.5)$$

Now we choose fixed elements

$$\sigma_q = \sigma_{q'} \in \bigcap_{q, q' \neq p \in \overline{1, 2m}} \ker_{\varphi_p|_{\Gamma_1}} \setminus (\ker_{\varphi_q|_{\Gamma_1}} \cup \ker_{\varphi_{q'}|_{\Gamma_1}}) \quad \text{for } q \in \mathcal{U}_1 \quad (5.6)$$

(cf. (4.22)) and

$$\sigma_n \in \Gamma_2. \quad (5.7)$$

Up to equivalence of the following construction of the Lie algebras of type K, we can assume

$$\varphi_p(\sigma_p) = -1 \quad \text{for } p \in \mathcal{U}_1. \quad (5.8)$$

For convenience, we let

$$\sigma_p = 0 \quad \text{for } p \in \mathcal{U}_2. \quad (5.9)$$

Note by the assumption (5.1),

$$\mathcal{J}_q = \mathbb{N} \quad \text{for } q \in \mathcal{U}_2. \quad (5.10)$$

We define an operator ∂ on \mathcal{A} by

$$\partial(x^{\alpha, \vec{i}}) = \left(\sum_{p \in \mathcal{U}_1} \varphi_p(\alpha) + \sum_{q \in \mathcal{U}_2} i_q \right) x^{\alpha, \vec{i}} \quad \text{for } (\alpha, \vec{i}) \in \Gamma \times \vec{\mathcal{J}}. \quad (5.11)$$

Then ∂ is a derivation of (\mathcal{A}, \cdot) . Moreover, we define an algebraic operation $[\cdot, \cdot]_K$ on \mathcal{A} by

$$[u, v]_K = \sum_{p=1}^m x_1^{\sigma_p} (\partial_p(u) \partial_{p'}(v) - \partial_{p'}(u) \partial_p(v)) + x_1^{\sigma_n} [(2 - \partial)(u) \partial_n(v) - \partial_n(u) (2 - \partial)(v)] \quad (5.12)$$

for $u, v \in \mathcal{A}$ (cf. (4.22)). It can be verified that $(\mathcal{A}, [\cdot, \cdot]_K)$ forms a Lie algebra. We call $(\mathcal{A}, [\cdot, \cdot]_K)$ a *generalized Lie algebra of Contact type*.

Theorem 5.1. *The pair $(\mathcal{A}, [\cdot, \cdot]_K)$ forms a simple Lie algebra.*

Proof. Let I be a nonzero ideal of \mathcal{A} . Set

$$\mathcal{A}' = \text{span} \{x^{\alpha, \vec{i}} \mid (\alpha, \vec{i}) \in \Gamma_1 \times \vec{\mathcal{J}}, i_n = 0\}. \quad (5.13)$$

In fact,

$$\mathcal{A}' = \{u \in \mathcal{A} \mid \partial_n(u) = 0\}. \quad (5.14)$$

Step 1. $I \cap \mathcal{A}' \neq \{0\}$.

We set

$$\mathcal{A}_{[k]} = \text{span} \{x^{\alpha, \vec{i}} \mid (\alpha, \vec{i}) \in \Gamma \times \vec{\mathcal{J}}, i_n \leq k\} \quad \text{for } k \in \mathbb{N}; \quad \mathcal{A}_{[-1]} = \emptyset. \quad (5.15)$$

Define

$$\hat{k} = \min\{k \mid \mathcal{A}_{[k]} \cap I \neq \{0\}\}. \quad (5.16)$$

Let $0 \neq u \in \mathcal{A}_{[\hat{k}]} \cap I$. We write

$$u = u_0 + u' \quad \text{with } u' \in \mathcal{A}_{[\hat{k}-1]} \quad (5.17)$$

and

$$u_0 = \sum_{(\alpha, \vec{i}) \in \Gamma \times \vec{\mathcal{J}}, i_n = \hat{k}} a_{\alpha, \vec{i}} x^{\alpha, \vec{i}}, \quad a_{\alpha, \vec{i}} \in \mathbb{F}. \quad (5.18)$$

Moreover, we define

$$\hat{i}(u) = |\{\alpha \mid a_{\alpha, \vec{i}} \neq 0\}|. \quad (5.19)$$

Let

$$\hat{i} = \min\{\hat{i}(v) \mid 0 \neq v \in \mathcal{A}_{[\hat{k}]} \cap I\}. \quad (5.20)$$

Let $0 \neq u \in \mathcal{A}_{[\hat{k}]} \cap I$ such that $\hat{i}(u) = \hat{i}$. Write u as (5.17) and (5.18).

Case 1. There exist $a_{\alpha, \vec{i}} \neq 0$ and $a_{\beta, \vec{\mathcal{J}}} \neq 0$ such that $\alpha \in \Gamma_1$ and $\beta \notin \Gamma_1$.

Note that

$$[1, x^{\gamma, \vec{l}}]_K = 2\varphi_n(\gamma)x^{\gamma+\sigma_n, \vec{l}} + 2l_n x^{\gamma+\sigma_n, \vec{l}-1_{[n]}} \quad (5.21)$$

for $(\gamma, \vec{l}) \in \Gamma \times \vec{\mathcal{J}}$ by (5.11) and (5.12). In particular, we have

$$[1, x^{\alpha, \vec{i}}]_K \equiv 0, \quad [1, x^{\beta, \vec{\mathcal{J}}}]_K \equiv 2\varphi_n(\beta)x^{\beta+\sigma_n, \vec{\mathcal{J}}} \pmod{\mathcal{A}_{[\hat{k}-1]}}. \quad (5.22)$$

Hence we have

$$0 \neq [1, u]_K \in \mathcal{A}_{[\hat{k}]} \cap I, \quad 0 < \hat{i}([1, u]_K) < \hat{i}(u) = \hat{i}, \quad (5.23)$$

which contradicts (5.20).

Case 2. $\alpha \in \Gamma_1$ whenever $a_{\alpha, \vec{i}} \neq 0$.

If $\hat{k} = 0$, then $u \in \mathcal{A}'$. So $I \cap \mathcal{A}' \neq \{0\}$. Assume that $\hat{k} > 0$. By (5.21), $0 \neq [1, u]_K \in \mathcal{A}_{[\hat{k}-1]} \cap I$, which contradicts (5.16).

Case 3. There exists $a_{\alpha, \vec{i}} \neq 0$ such that

$$(\vartheta(\alpha, \vec{i}) + 2)\varphi_n(\alpha) + \vartheta(\alpha, \vec{i})\varphi_n(\sigma_n) \neq 0, \quad (5.24)$$

where

$$\vartheta(\alpha, \vec{i}) = 2 - \sum_{p \in \mathcal{U}_1} \varphi_p(\alpha) - \sum_{q \in \mathcal{U}_2} i_q. \quad (5.25)$$

Note that

$$[x_1^\kappa, x^{\gamma, \vec{l}}]_K = (2\varphi_n(\gamma) - \varphi_n(\kappa)\vartheta(\gamma, \vec{l}))x^{\gamma+\kappa+\sigma_n, \vec{l}} + 2l_n x^{\gamma+\kappa+\sigma_n, \vec{l}-1_{[n]}} \quad (5.26)$$

for $\kappa \in \Gamma_2$, $\gamma \in \Gamma$ and $\vec{l} \in \vec{\mathcal{J}}$. Moreover, we write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_r \in \Gamma_r$ by (5.2) and have

$$0 \neq [x_1^{-\alpha_2-\sigma_n}, u] \in \mathcal{A}_{[\hat{k}]} \cap I, \quad 0 < \hat{i}([x_1^{-\alpha_2-\sigma_n}, u]_K) \leq \hat{i} \quad (5.27)$$

and $[x_1^{-\alpha_2 - \sigma_n}, u]$ contains the following term:

$$a_{\alpha, \vec{i}}((\vartheta(\alpha, \vec{i}) + 2)\varphi_n(\alpha) + \vartheta(\alpha, \vec{i})\varphi_n(\sigma_n))x^{\alpha_1, \vec{i}}. \quad (5.28)$$

Replacing u by $[x_1^{-\alpha_2 - \sigma_n}, u]$, we go back to Cases 1 and 2.

Case 4.

$$\alpha \notin \Gamma_1, \quad (\vartheta(\alpha, \vec{i}) + 2)\varphi_n(\alpha) + \vartheta(\alpha, \vec{i})\varphi_n(\sigma_n) = 0 \quad \text{whenever} \quad a_{\alpha, \vec{i}} \neq 0. \quad (5.29)$$

In this case, $\Gamma_2 \neq \{0\}$.

Subcase 1. There exists $a_{\alpha, \vec{i}} \neq 0$ such that $\vartheta(\alpha, \vec{i}) \neq -2$.

We can choose $-\sigma_n \neq \gamma \in \Gamma_2$ such that

$$2\varphi_n(\alpha) \neq \varphi_n(\gamma)\vartheta(\alpha, \vec{i}) \quad (5.30)$$

because $|\Gamma_2| = \infty$. Then $[x_1^\gamma, u]$ contains the following term

$$a_{\alpha, \vec{i}}(2\varphi_n(\alpha) - \varphi_n(\gamma)\vartheta(\alpha, \vec{i}))x^{\alpha + \gamma + \sigma_n, \vec{i}}. \quad (5.31)$$

Thus, replacing u by $[x_1^\gamma, u]$, we go back to Case 3 because $\varphi_n|_{\Gamma_2}$ is injective.

Subcase 2. $\vartheta(\alpha, \vec{i}) = -2$ whenever $a_{\alpha, \vec{i}} \neq 0$.

In this subcase, $\sigma_n = 0$ by (5.29). Moreover, (5.21) becomes

$$[1, x^{\gamma, \vec{l}}]_K = 2\partial_n(x^{\gamma, \vec{l}}) = 2\varphi_n(\gamma)x^{\gamma, \vec{l}} + 2l_n x^{\gamma, \vec{l}-1_{[n]}} \quad (5.32)$$

for $(\gamma, \vec{l}) \in \Gamma \times \vec{\mathcal{J}}$. For any $\beta \in \Gamma_2$, we let

$$\mathcal{A}_{[\beta]} = \text{span} \{x^{\alpha + \beta, \vec{i}} \mid (\alpha, \vec{i}) \in \Gamma_1 \times \vec{\mathcal{J}}\}. \quad (5.33)$$

Then we have:

$$\mathcal{A} = \bigoplus_{\beta \in \Gamma_2} \mathcal{A}_{[\beta]}, \quad \mathcal{A}_{[\beta]} = \{u \in \mathcal{A} \mid (\text{ad}_1 - 2\varphi_n(\beta))^j(u) = 0 \text{ for some } j \in \mathbb{N}\} \quad (5.34)$$

by (5.32) and (5.33).

Let $a_{\alpha, \vec{i}} \neq 0$ be such that $|\vec{i}|$ is maximal (cf. (4.14)). Again we write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_r \in \Gamma_r$. If $\mathcal{J}_1 = \mathbb{N}$, then we have

$$[x^{\alpha, \vec{i}}, x^{-\alpha_2, 1_{[1]}}]_K = \varphi_n(\alpha)x^{\alpha_1, \vec{i}+1_{[1]}} - \varphi_{m+1}(\alpha)x^{\alpha_1, \vec{i}} - i_1 x^{\alpha, \vec{i}-1_{[1]}} \quad (5.35)$$

by (5.11), (5.12) and (5.29). Thus $[u, x^{-\alpha_2, 1_{[1]}}]$ contains the following term:

$$a_{\alpha, \vec{i}}\varphi_n(\alpha)x^{\alpha_1, \vec{i}+1_{[1]}} \in \mathcal{A}_{0_{\Gamma_2}}. \quad (5.36)$$

Since $[x^{-\alpha_2, 1_{[1]}}, u]_K \in I$, (5.34) and (5.36) imply

$$I \cap \mathcal{A}_{0_{\Gamma_2}} \neq \{0\}. \quad (5.37)$$

Pick any $0 \neq v \in I \cap \mathcal{A}_{0_{\Gamma_2}}$. Applying ad_1 on v repeatedly, we can get a nonzero element $w \in \mathcal{A}'$ by (5.32).

If $\mathcal{J}_1 = \{0\}$, then $\varphi_1 \not\equiv 0$. So $\sigma_1 \neq 0$ by the assumption in (5.6). Note

$$[x_1^{\sigma_1 - \alpha_2}, x^{\alpha, \vec{i}}]_K \equiv (\varphi_1(\alpha) - \varphi_{m+1}(\alpha))x^{\alpha_1 + 2\sigma_1, \vec{i}} + \varphi_n(\alpha)x^{\alpha_1 + \sigma_1, \vec{i}} \pmod{\mathcal{A}_{[\hat{k}-1]}} \quad (5.38)$$

by (5.8), (5.11), (5.12) and (5.29). Thus $[x_1^{\sigma_1 - \alpha_2}, u]_K$ contains the following term

$$a_{\alpha, \vec{i}} \varphi_n(\alpha) x^{\alpha_1 + \sigma_1, \vec{i}} \in \mathcal{A}_{0_{\Gamma_2}}. \quad (5.39)$$

Since $[x_1^{\sigma_1 - \alpha_2}, u]_K \in I$, (5.34) and (5.39) imply (5.37). Thus $I \cap \mathcal{A}' \neq \{0\}$.

Step 2. $1 \in I$.

Note that for $u, v \in \mathcal{A}'$, we have

$$[u, v]_K = \sum_{p=1}^m x_1^{\sigma_p} [\partial_p(u) \partial_{m+p}(v) - \partial_{m+p}(u) \partial_p(v)] \quad (5.40)$$

by (5.11) and (5.12). Moreover, (5.40) is a special case of (4.12) with $m_1 = m$, $\phi \equiv 0$ and \mathcal{A} replaced by \mathcal{A}' . By the proof of Theorem 4.2, we have

$$I \cap \mathcal{A}' = \mathbb{F} \quad (5.41)$$

or

$$I \cap \mathcal{A}' = \mathcal{A}' \quad \text{if } \mathcal{J}_p = \mathbb{N} \text{ for some } p \in \overline{1, 2m} \quad (5.42)$$

or

$$I \cap \mathcal{A}' = \text{span} \{x_1^\alpha \mid \sum_{p=1}^m \sigma_p \neq \alpha \in \Gamma_1\} \quad \text{if } \mathcal{J}_p = \{0\} \text{ for any } p \in \overline{1, 2m} \quad (5.43)$$

(cf. (4.70), (4.79)). Note by (5.1), (5.3) and (5.6),

$$\sum_{p=1}^m \sigma_p \neq 0 \quad (5.44)$$

in the third case. Thus we always have $1 \in I$ in all the three cases (5.41)-(5.43).

Step 3. $I = \mathcal{A}$ from $1 \in I$.

By (5.21) and induction on l_n , we can prove $I = \mathcal{A}$ if $\mathcal{J}_n = \mathbb{N}$. Assume $\mathcal{J}_n = \{0\}$. Then $\varphi_n \neq 0$ by the assumptions in (5.1). Note that (5.21) shows

$$x^{\alpha_1 + \alpha_2, \vec{i}} \in I \quad \text{for } \alpha_1 \in \Gamma_1, \sigma_n \neq \alpha_2 \in \Gamma_2, \vec{i} \in \vec{\mathcal{J}}. \quad (5.45)$$

For any $0, \sigma_n \neq \kappa \in \Gamma_2$ and $(\beta, \vec{\mathcal{J}}) \in \Gamma_1 \times \vec{\mathcal{J}}$ such that $\vartheta(\beta, \vec{\mathcal{J}}) \neq -2$, we have

$$x^{\beta + \sigma_n, \vec{\mathcal{J}}} = [\varphi_n(\kappa)(2 + \vartheta(\beta, \vec{\mathcal{J}}))]^{-1} [x_1^{-\kappa}, x^{\beta + \kappa, \vec{\mathcal{J}}}]_K \in I. \quad (5.46)$$

Since $\vartheta(0, 1_{[1]}) = 1$ if $\mathcal{J}_1 = \mathbb{N}$ and $\vartheta(\sigma_1, \vec{0}) = 3$ if $\varphi_1 \neq 0$ by (5.8) and (5.25), (5.45) and (5.46) imply either $x_2^{1_{[1]}} \in I$ or $x_1^{\sigma_1} \in I$. Moreover, the arguments in Cases 1 and 2 of Step 2 in the proof of Theorem 4.2 show that $I = \mathcal{A}$ if $\vec{\mathcal{J}} \neq \{\vec{0}\}$ and

$$x_1^{\alpha_1 + \alpha_2} \in I \quad \text{for } \sum_{p=1}^m \sigma_p \neq \alpha_1 \in \Gamma_1, \alpha_2 \in \Gamma_2 \quad (5.47)$$

if $\vec{\mathcal{J}} = \{\vec{0}\}$. When $\vec{\mathcal{J}} = \{\vec{0}\}$, (5.45), (5.46) and (5.47) imply $I = \mathcal{A}$ because $\vartheta(\sigma, \vec{0}) = 2(m+1) \neq -2$ by (5.8) and (5.25). \square

Example. Suppose that we have picked (2.5). Let k be the number of $\mathcal{J}_p = \{0\}$ with $p \in \overline{1, n}$. Pick an integer ℓ such that $k \leq \ell \leq n$. We define

$$\zeta_p(\alpha_1, \dots, \alpha_\ell) = \alpha_p \quad \text{for } p \in \overline{1, \ell}, (\alpha_1, \alpha_2, \dots, \alpha_\ell) \in \mathbb{F}^\ell \quad (5.48)$$

and

$$\zeta_q \equiv 0 \quad \text{for } q \in \overline{\ell+1, n}. \quad (5.49)$$

Pick any permutation ι on $\overline{1, n}$ such that

$$\iota(p) \leq \ell \text{ if } \mathcal{J}_p = \{0\} \text{ with } p \in \overline{1, n}. \quad (5.50)$$

Moreover, we can assume $\iota(n) = \ell$ if $\iota(n) \leq \ell$. Take Γ to be an additive subgroup of \mathbb{F}^ℓ such that $\Gamma \supset \mathbb{Z}^\ell$ and

$$\Gamma \supset \{(0, \dots, 0, \alpha_\ell) \mid (\alpha_1, \dots, \alpha_\ell) \in \Gamma\} \text{ if } \iota(n) = \ell. \quad (5.51)$$

We let

$$\varphi_p = \zeta_{\iota(p)}|_\Gamma \quad \text{for } p \in \overline{1, n}. \quad (5.52)$$

Then (5.1)-(5.3) and (5.5) hold.

In particular, we can take (2.34) and (2.35).

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